From Pascal to Calculus

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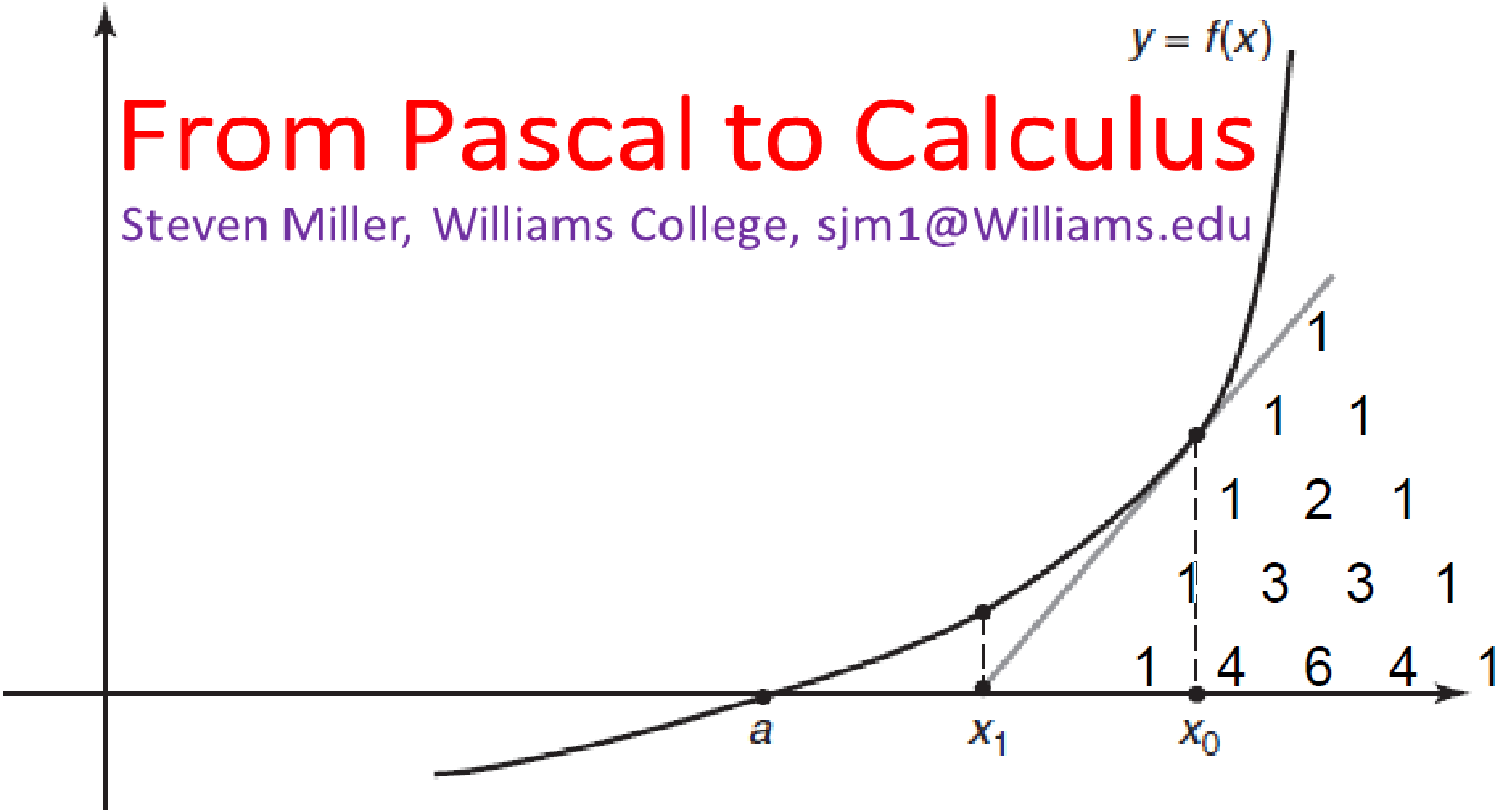

PART I:
Pascal's Triangle

Goal of the Talk

We will go from Algebra to one of the two main parts of calculus: differentiation.

Differentiation is all about how functions change.

We will review functions, discuss limits, find derivatives, and see a wonderful application: Newton's Method to numerically approximate $\sqrt{3}$. We will then extend to other roots, and see chaos and fractals.

Pascal's Triangle

The numbers in the n^{th} row of Pascal's Triangle are the coefficients we obtain in expanding $(x+y)^n$.

Equivalently, we have two diagonals of 1, and all other elements are the sum of the elements in the row above immediately to the left and immediately to the right.

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

FOIL

FOIL stands for FIRST, OUTSIDE, INSIDE and LAST.

It provides a framework to multiply (a+b) and (c+d).

We have:

$$(a + b) * (c + d) = \underbrace{a * c}_{\text{FIRST}} + \underbrace{a * d}_{\text{OUTSIDE}} + \underbrace{b * c}_{\text{INSIDE}} + \underbrace{b * d}_{\text{LAST}}.$$

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We have:

$(a + b) * (c + d) = a * c + a * d + b * c + b * d.$

Thus:

$(3 + 5) * (7-2) = 3 * 7 + 3 * (-2) + 5 * 7 + 5 * (-2) = 21 – 6 + 35 – 10 = 40$ (which is $8 * 5$).

$(x + y) * (x – y) = x * x + x * (-y) + y * x + y * (-y) = x^2 – x y + y x – y^2 = x^2 – y^2.$

$(x + y) * (x + y) = x * x + x * y + y * x + y * y = x^2 + x y + y x + y^2 = x^2 + 2 x y + y^2.$

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We can repeatedly apply it, and its generalizations…..

We have:

$(x + y)^2 = (x + y) * (x + y) = x * x + x * y + y * x + y * y = x^2 + x y + y x + y^2 = x^2 + 2 x y + y^2.$

So:

$(x + y)^3 = (x + y) * (x + y)^2 = (x + y) * (x^2 + 2 x y + y^2)$

$= x * (x^2 + 2 x y + y^2) + y * (x^2 + 2 x y + y^2)$

$= (x^3 + 2 x^2 y + x y^2) + (x^2 y + 2 x y^2 + y^3)$

$= x^3 + 3 x^2 y + 3 x y^2 + y^3.$

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We can repeatedly apply it, and its generalizations.....

We have:

$(x + y)^2 = (x + y) * (x + y) = x * x + x * y + y * x + y * y = x^2 + x\,y + y\,x + y^2 = \mathbf{1}\,x^2 + \mathbf{2}\,x\,y + \mathbf{1}\,y^2.$

So:

$(x + y)^3 = (x + y) * (x + y)^2 = (x + y) * (x^2 + 2\,x\,y + y^2)$

$= x * (x^2 + 2\,x\,y + y^2) + y * (x^2 + 2\,x\,y + y^2)$

$= (x^3 + 2\,x^2\,y + x\,y^2) + (x^2\,y + 2\,x\,y^2 + y^3)$

$= \mathbf{1}\,x^3 + \mathbf{3}\,x^2\,y + \mathbf{3}\,x\,y^2 + \mathbf{1}\,y^3.$

Expanding $(x + y)^n$

$(x + y)^1 = \mathbf{1}\, x + \mathbf{1}\, y$

$(x + y)^2 = \mathbf{1}\, x^2 + \mathbf{2}\, x y + \mathbf{1}\, y^2.$

$(x + y)^3 = \mathbf{1}\, x^3 + \mathbf{3}\, x^2 y + \mathbf{3}\, x y^2 + \mathbf{1}\, y^3.$

This is the start of Pascal's Triangle…..

How should we define $(x + y)^0$? Well, we often say things to to zeroth power are 1, so we extend to….

Expanding $(x + y)^n$

$(x + y)^0 = \mathbf{1}$

$(x + y)^1 = \mathbf{1}\, x + \mathbf{1}\, y$

$(x + y)^2 = \mathbf{1}\, x^2 + \mathbf{2}\, x\, y + \mathbf{1}\, y^2.$

$(x + y)^3 = \mathbf{1}\, x^3 + \mathbf{3}\, x^2\, y + \mathbf{3}\, x\, y^2 + \mathbf{1}\, y^3.$

This is the start of Pascal’s Triangle…..

We re-write it in triangular form….

1

1 1

1 2 1

1 3 3 1

Expanding $(x + y)^n$

$(x + y)^0 = \mathbf{1}$

$(x + y)^1 = \mathbf{1}\,x + \mathbf{1}\,y$

$(x + y)^2 = \mathbf{1}\,x^2 + \mathbf{2}\,x\,y + \mathbf{1}\,y^2$.

$(x + y)^3 = \mathbf{1}\,x^3 + \mathbf{3}\,x^2\,y + \mathbf{3}\,x\,y^2 + \mathbf{1}\,y^3$.

We can keep going and get more and more rows…..

1

1 1

1 2 1

1 3 3 1

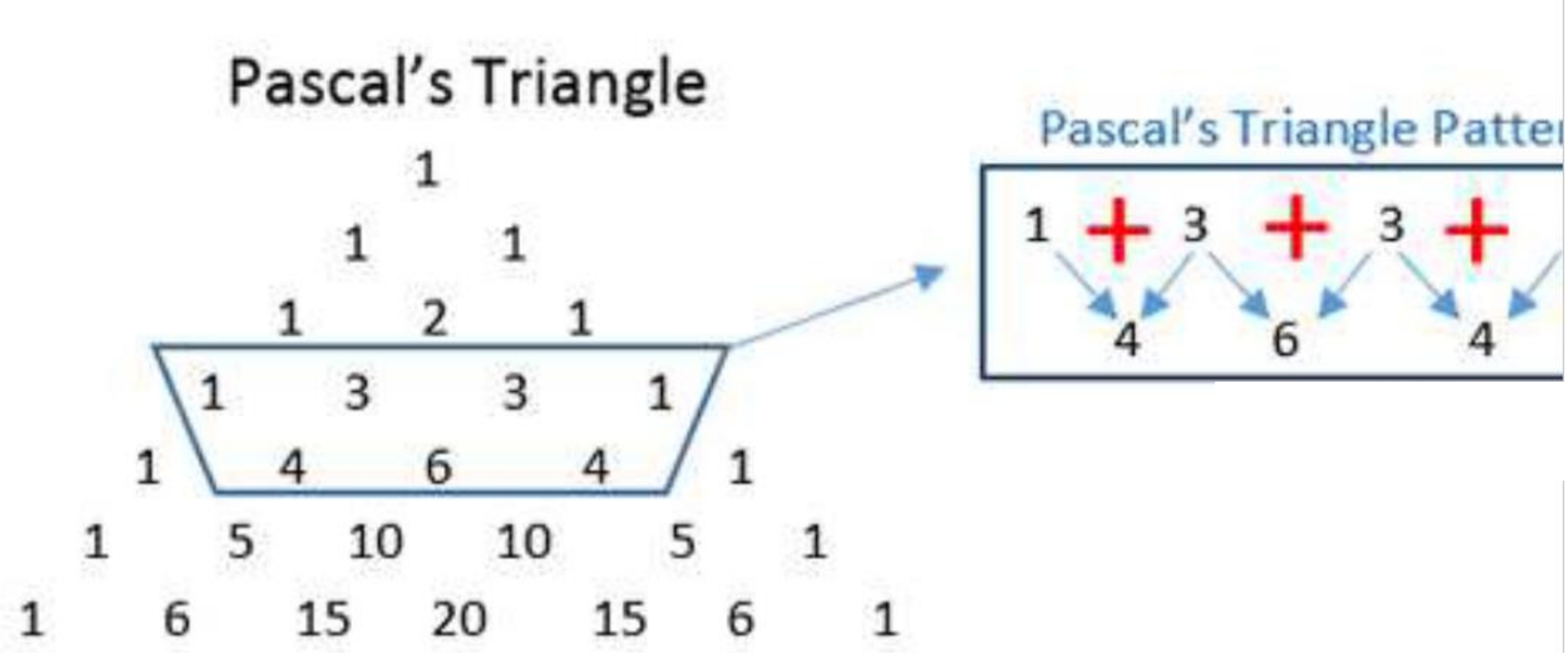
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Why is the Pascal Relation true? Each number is the sum of what is immediately above to the right and to the left.



Sketch of the proof:

Assume we know one row, say
$(x+y)^5 = x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5$.

Then

$$(x+y)^6 = (x+y)\,(x+y)^5$$

$$= x\,(x+y)^5 + y\,(x+y)^5$$

$$= x\,(x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5) + y\,(x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5)$$

$$= (x^6 + 5\,x^5\,y + 10\,x^4\,y^2 + 10\,x^3\,y^3 + 5\,x^2\,y^4 + x\,y^5) + (x^5\,y + 5\,x^4\,y^2 + 10\,x^3\,y^3 + 10\,x^2\,y^4 + 5\,x\,y^5 + y^6)$$

$$= x^6 + 5\,x^5\,y + 10\,x^4\,y^2 + 10\,x^3\,y^3 + 5\,x^2\,y^4 + x\,y^5$$
$$+ x^5\,y + 5\,x^4\,y^2 + 10\,x^3\,y^3 + 10\,x^2\,y^4 + 5\,x\,y^5 + y^6$$

$$= x^6 + (5+1)\,x^5\,y + (10+5)\,x^4\,y^2 + (10+10)\,x^3\,y^3 + (5+10)\,x^2\,y^4 + (1+5)\,x\,y^5 + y^6$$

$$= x^6 + 6\,x^5\,y + 15\,x^4\,y^2 + 20\,x^3\,y^3 + 15\,x^2\,y^4 + 6\,x\,y^5 + y^6$$

Pascal's Triangle

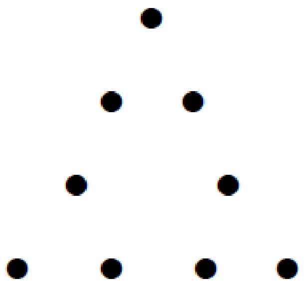
While we can prove many properties of the coefficients of Pascal's triangle, for small n we can just expand directly.

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	
1		7		21		35		35		21		7		1

Pascal’s Triangle

Modify Pascal’s triangle: • if odd, blank if even.

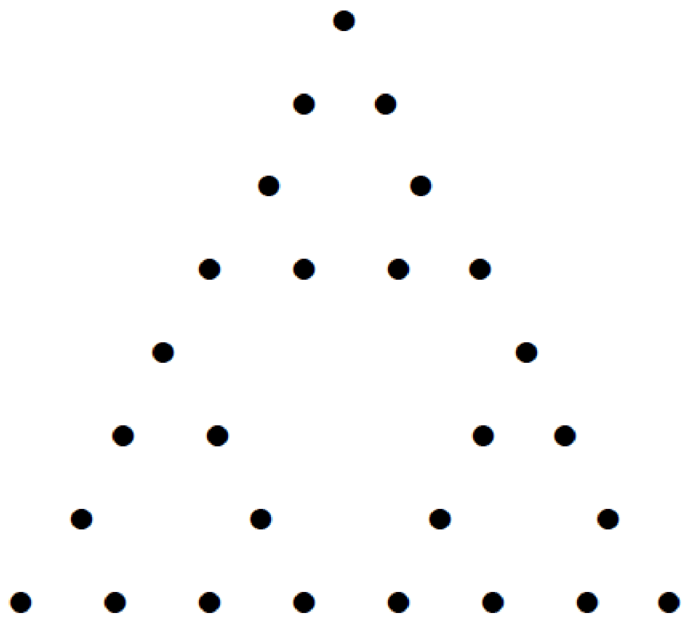
If we have just one row we would see •, if we have four rows we would see



Pascal's Triangle

Modify Pascal's triangle: • if odd, blank if even.

For eight rows we find



Pascal's Triangle

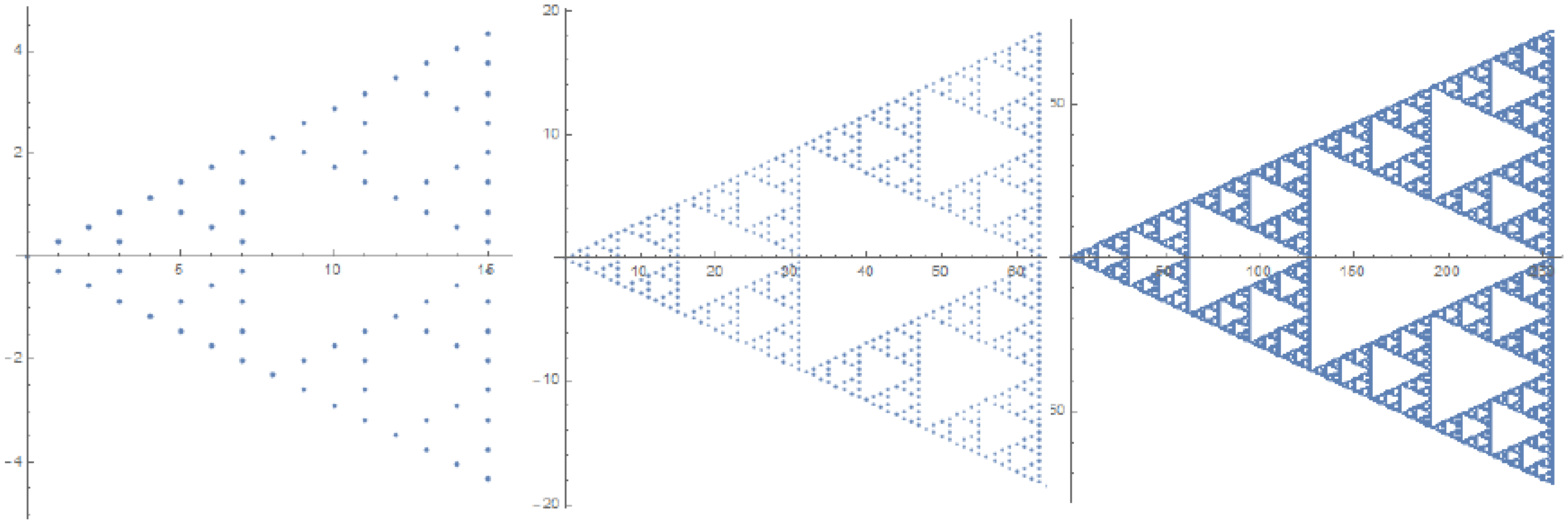


Figure: Plot of Pascal's triangle modulo 2 for 2^4, 2^8 and 2^{10} rows.

https://www.youtube.com/watch?v=tt4_4YajqRM (start 1:35)

PART II:
Algebra and Limits

Evaluating Functions

A function takes an input and sends it to an output.

We often use the letter f to denote the function, and put the input in parentheses.

A linear function is of the form $f(x) = a x + b$ for fixed constants a and b.

For example: $f(x) = 3x – 5$ or $f(x) = 7x + 2$ or $f(x) = 4x + 17$.

Evaluating Functions

For example: $f(x) = 3x - 5$. Let's evaluate it at a few choices of x.

We have:

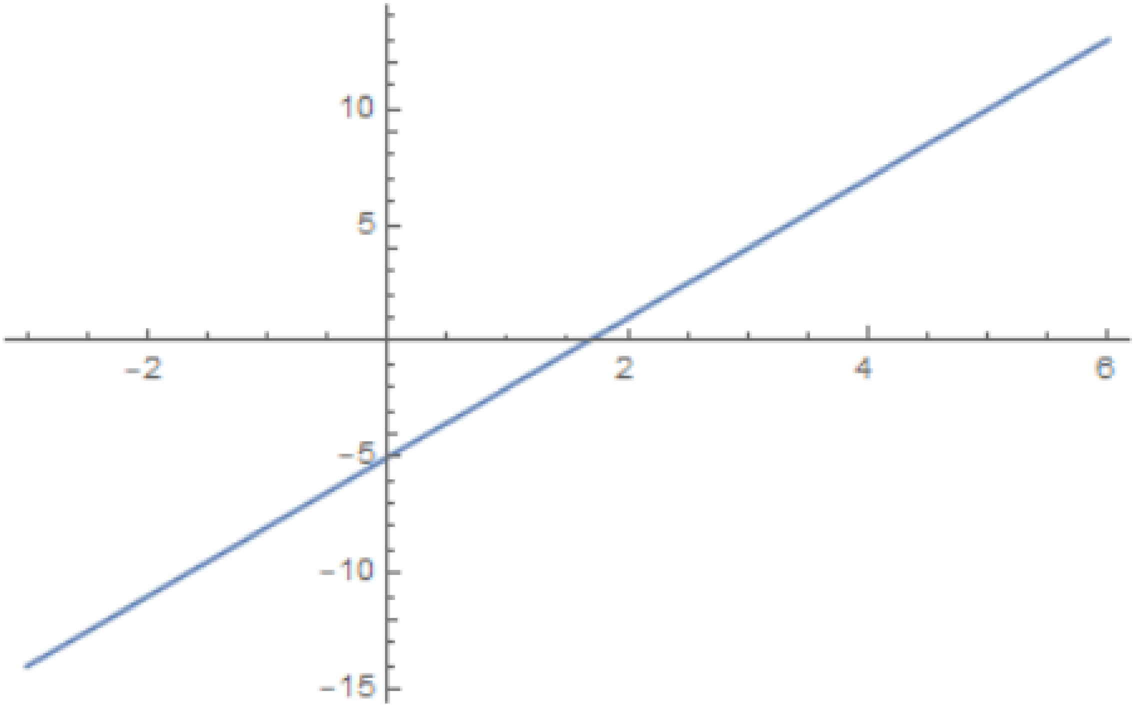
$f(0) = 3 * 0 - 5 = -5$

$f(1) = 3 * 1 - 5 = -2$

$f(2) = 3 * 2 - 5 = 1$

$f(3) = 3 * 3 - 5 = 4$

$f(4) = 3 * 4 - 5 = 7$



Quadratic Functions

Quadratic functions of the form $f(x) = ax^2 + bx + c$ for constants a, b, c.

Consider $f(x) = 2x^2 - 3x + 4$.

We have:

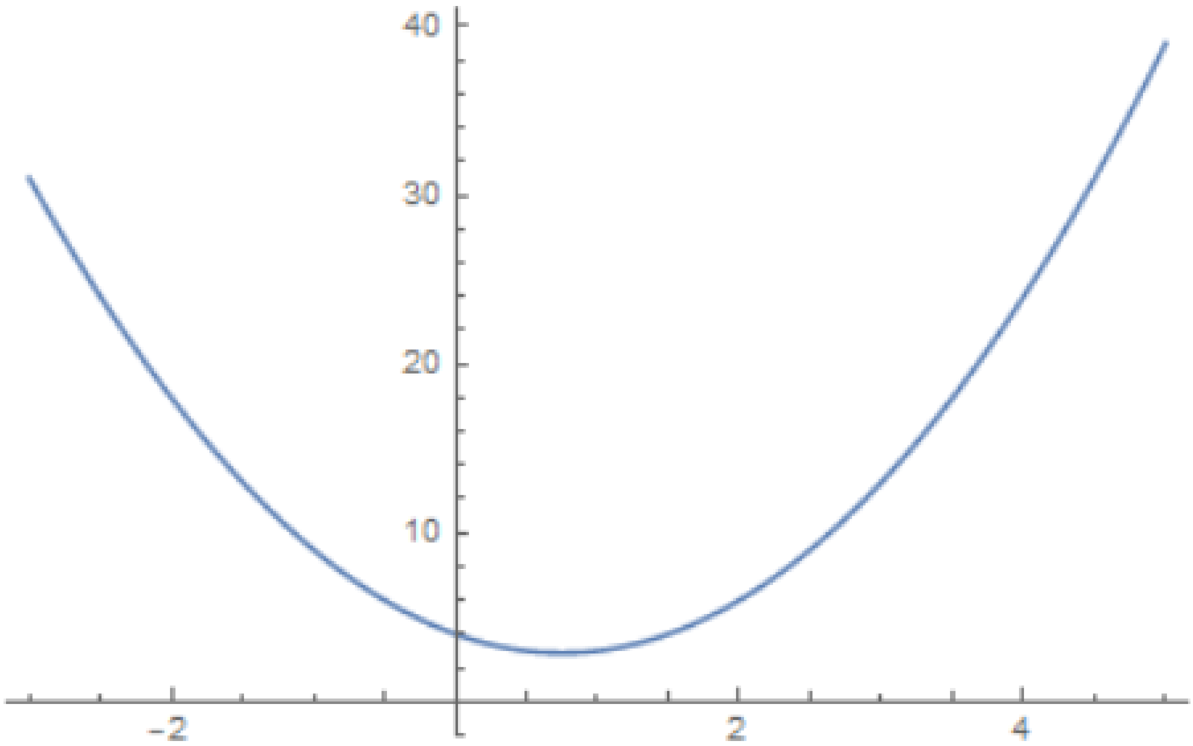
$f(0) = 2 * 0^2 - 3 * 0 + 4 = 4$

$f(1) = 2 * 1^2 - 3 * 1 + 4 = 3$

$f(2) = 2 * 2^2 - 3 * 2 + 4 = 6$

$f(3) = 2 * 3^2 - 3 * 3 + 4 = 13$

$f(4) = 2 * 4^2 - 3 * 4 + 4 = 24$



Polynomials

More generally can look at a polynomial of degree n: have constants a_n, a_{n-1}, …, a_1, a_0 so that $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$.

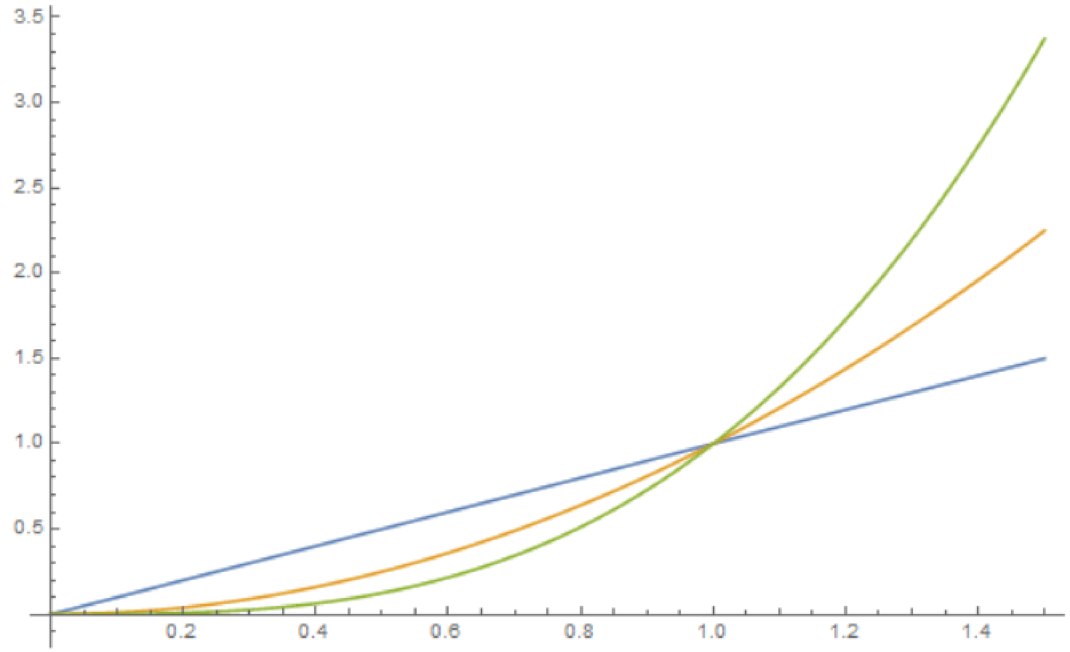
Here is a plot of

$f(x) = x$ (linear)

$g(x) = x^2$ (quadratic)

$h(x) = x^3$ (cubic)

for x between 0 and 1.5.



Limits

One of the most important concepts in calculus is that of a limit.

We want to know what happens to the output of a function as the inputs approach a specific value.

For polynomials the limit is easy. If f(x) = 3 x + 5, what is the limit of f(x) as x approaches 2? It would just be $\lim_{x \to 2} f(x) =$

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Another way of writing x approaches 2 is to write x as $2 + h$, and take the limit as h goes to 0.

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$\lim_{x \to 2} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0}$???

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$\lim_{x \to 2} f(x) = \lim_{h \to 0} f(2+h) = \lim_{h \to 0} (3 * (2+h) + 5) = \lim_{h \to 0} (6 + 3\, h + 5)$.

Now the limit of the sum is the sum of the limits, and we have

$\lim_{h \to 0} (6 + 3\, h + 5) = \lim_{h \to 0} 6 + \lim_{h \to 0} 3\, h + \lim_{h \to 0} 5 =$

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More on Limits

When you compute a limit, say the limit as x approaches 2, we can write x as 2 + h and you should think of h as a very small number that is NOT zero.

We are talking about the limit as h approaches 0, but it is never 0.

Consider $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. What will this equal?

More on Limits

We are talking about the limit as h approaches 0, but it is never 0.

Consider $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$. We write x as 2 + h, and note from FOIL that

$(2+h)^2 = 2*2 + 2\ h + h\ 2 + h^2 = 4 + 4h + h^2$. We must be careful as, at 2, have 0/0.

We have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{h \to 0} \frac{???}{???}$$

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We have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{h \to 0} \frac{(2+h)^2 - 4}{(2+h) - 2} = \lim_{h \to 0} \frac{4+4h+h^2 - 4}{2+h - 2} = \lim_{h \to 0} \frac{4h+h^2}{h} = \lim_{h \to 0} (4 + h) = 2.$$

More on Limits

We have

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{h\to 0} \frac{(2+h)^2-4}{(2+h)-2} = \lim_{h\to 0} \frac{4+4h+h^2-4}{2+h-2} = \lim_{h\to 0} \frac{4h+h^2}{h} = \lim_{h\to 0} (4+h) = 4.$$

Could have noticed $x^2 - 4 = (x-2)(x+2)$ and cancel the x-2:

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} (x+2) = 2+2 = 4.$$

Not all limits are easy to compute. In an earlier lecture came up with some formulas for π….

$$\text{NumSides} = 16,\ \text{SemiPerim} = \frac{8}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)},\ \text{or about } 3.12145$$

$$\text{NumSides} = 32,\ \text{SemiPerim} = \frac{16}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)},\ \text{or about } 3.13655$$

$$\text{NumSides} = 64,\ \text{SemiPerim} = \frac{32}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)},\ \text{or about } 3.14033$$

$$\text{NumSides} = 128,\ \text{SemiPerim} = \frac{64}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)},\ \text{or about } 3.14128$$

PART III:
Introduction to Calculus: Differentiation

Average Speed

It is often a good idea to add units to a problem and tell a story. For example, if $y = f(x)$, maybe x represents time and f(x) distance.

Thus we might be plotting how far we are from home on a trip.

Let $f(0) = 0$ (we start at home) and end the trip at $x=2$, with $f(2) = 110$. What was our average speed? What was our fastest speed? Our slowest speed? Our most common speed?

Which of these questions can you answer?

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Which of these questions can you answer? Just the first: it is 110/2 = 55 (we should have units – maybe time is in hours and distance in miles, so 55 mph).

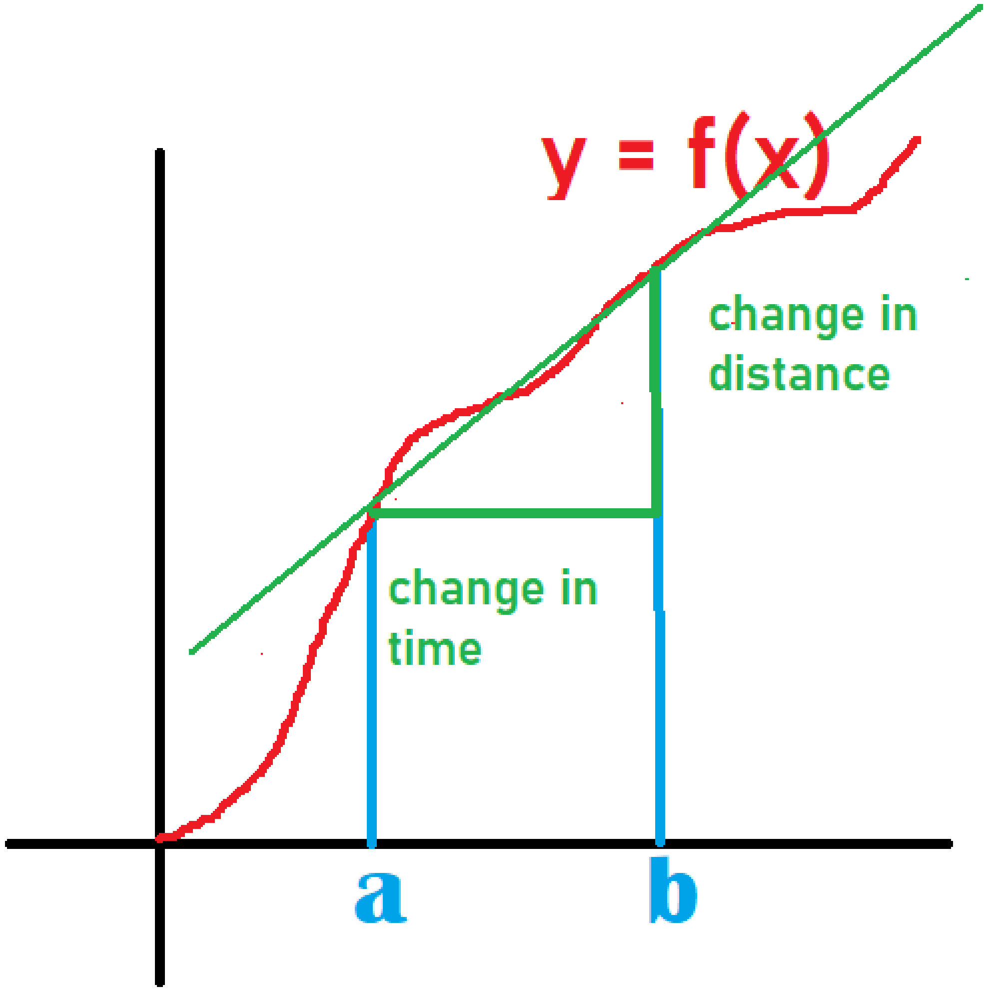
Computing Average Speed

It is easy to compute the average speed from time x=a to time x=b.

Let f(x) be our distance at time x. Then the average speed from x=a to x=b is just

$$Average\ Speed\ from\ a\ to\ b\ is\ \frac{f(b)-f(a)}{b-a}$$

This is the change in distance divided by the change in time.



Straight line is what the distance function would be if speed were constant and equal to the average speed from a to b.

Average Speed for a Linear Function

If f(x) = c x + d (a linear function) then the average speed is constant!

For example, say f(x) = 3x + 5.

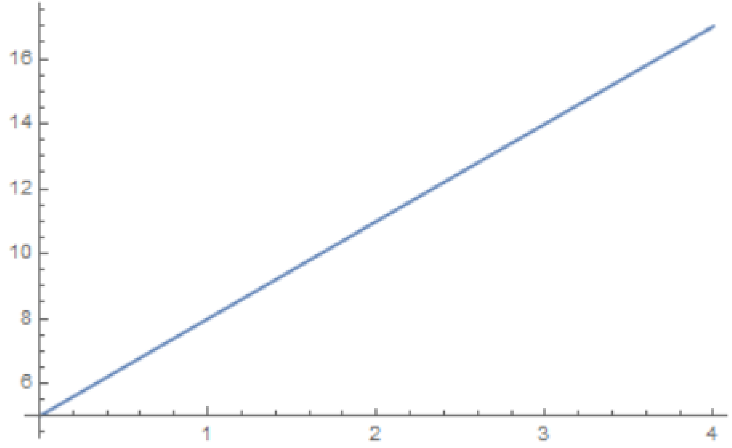
Let's compute the average speed from x=a to x=b.

Change in distance = f(b) – f(a) = (3b+5) – (3a+5) = 3b + 5 - 3a - 5 = 3b – 3a.

Change in time = b – a.

Average speed from x=a to x=b

is $\frac{3b-3a}{b-a} = \frac{3(b-a)}{b-a} = 3.$



Average Speed for a Quadratic Function

A quadratic function is more interesting.

Say $f(x) = x^2 + 3x + 1$, compute the average speed from x=a to x=b.

Change in distance is $f(b) – f(a) = (b^2 +3b+1) – (a^2+3a+1) = b^2+3b-a^2-3a$.

We can group: it equals $b^2-a^2 + 3b-3a = (b-a)(b+a) + 3(b-a) = (b-a)(b+a+3)$.

Change in time is b-a.

Thus average speed from x=a to x=b is

$$Average\ Speed\ from\ a\ to\ b = \frac{(b-a)(b+a+3)}{(b-a)} = b + a + 3.$$

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We can group: it equals $(b^2 - a^2)+ (3b-3a) = (b-a)(b+a) + 3(b-a) = (b-a)(b+a+3)$.

Change in time is b-a.

Thus average speed from x=a to x=b is

$$Average\ Speed\ from\ a\ to\ b = \frac{(b-a)(b+a+3)}{(b-a)} = b + a + 3.$$

What happens in the limit as b goes to a? What does this represent? What is this quantity equal to?

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We can group: it equals $b^2-a^2 + 3b-3a = (b-a)(b+a) + 3(b-a) = (b-a)(b+a+3)$.

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Thus average speed from x=a to x=b is

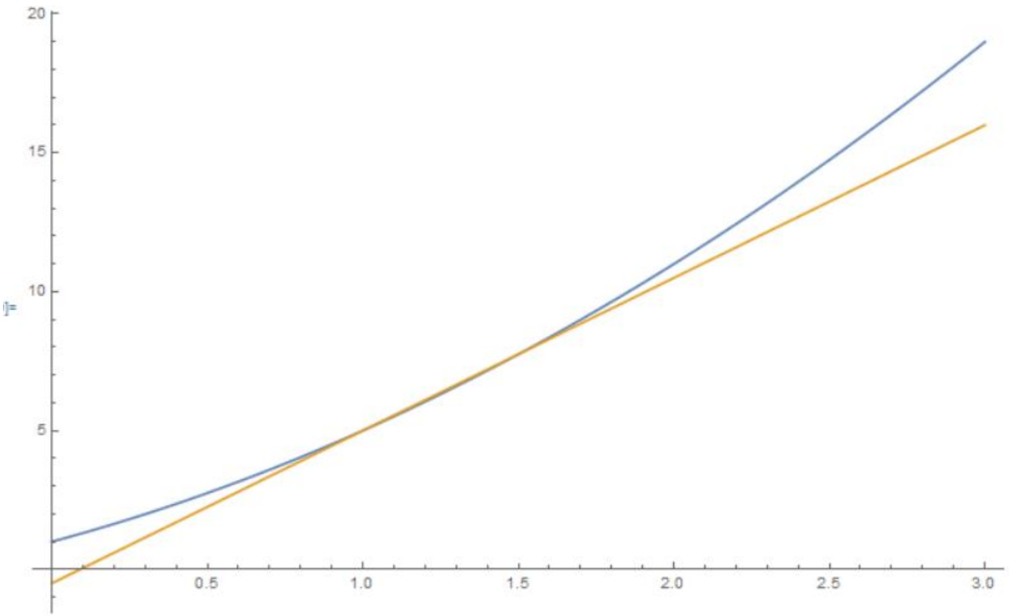
$$Average\ Speed\ from\ a\ to\ b = \frac{(b-a)(b+a+3)}{(b-a)} = b + a + 3.$$

What happens in the limit as b goes to a? What does this represent?
What is this quantity equal to? INSTANTANEOUS SPEED! Is 2a+3.

Calculating Average Speeds for $f(x) = x^2+3x+1$

We calculate the average speeds for f(x) from x=1 to x=b.

b	f(b)	Change distance	Change time	Ave Speed from x=1 to x=b
2.0	11.000	6.000	1.000	6.000
1.9	10.310	5.310	0.900	5.900
1.8	9.640	4.640	0.800	5.800
1.7	8.990	3.990	0.700	5.700
1.6	8.360	3.360	0.600	5.600
1.5	7.750	2.750	0.500	5.500
1.4	7.160	2.160	0.400	5.400
1.3	6.590	1.590	0.300	5.300
1.2	6.040	1.040	0.200	5.200
1.1	5.510	0.510	0.100	5.100
1.0	5.000	0.000	0.000	#DIV/0!



When we take b=1 the average speed calculation blows up. The plot on the right is with b = 1.5. Notice how the average speeds seem to converge to a number….

Instantaneous Speed

The instantaneous speed at x=a is the limit, if it exists, of the average speed from x=a to x=b as b converges to a:

$$Instantaneous\ Speed\ of\ f(x)\ at\ x = a\ is\ \lim_{b \to a} \frac{f(b)-f(a)}{b-a},$$

or equivalently

$$Instantaneous\ Speed\ at\ x = a\ is\ \lim_{h \to 0} \frac{f(a+h)-f(a)}{a+h-a} = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}.$$

Note this is a limit as h tends to 0, but h is never zero. Thus we do not have the undefined 0/0, we just have something arbitrarily close.

We denote this by f'(x), the prime indicates a NEW function related to the original function.

Instantaneous Speed for Linear Functions

Let's take f(x) = 3x + 5 and calculate the instantaneous speed at x=a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So $f'(a) = \lim_{h \to 0} \frac{(3(a+h)+5)-(3a+5)}{h}$ =

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So $f'(a) = \lim_{h \to 0} \frac{(3(a+h)+5)-(3a+5)}{h}$ = $\lim_{h \to 0} \frac{(3a+3h+5)-(3a+5)}{h}$ =

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So $f'(a) = \lim_{h \to 0} \frac{(3(a+h)+5)-(3a+5)}{h} = \lim_{h \to 0} \frac{(3a+3h+5)-(3a+5)}{h} = \lim_{h \to 0} \frac{3h}{h}$.

What is $\lim_{h \to 0} \frac{3h}{h}$? It is

Instantaneous Speed for Linear Functions

Let's take f(x) = 3x + 5 and calculate the instantaneous speed at x=a.

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So $f'(a) = \lim_{h \to 0} \frac{(3(a+h)+5)-(3a+5)}{h}$ = $\lim_{h \to 0} \frac{(3a+3h+5)-(3a+5)}{h}$ = $\lim_{h \to 0} \frac{3h}{h}$.

What is $\lim_{h \to 0} \frac{3h}{h}$? It is $\lim_{h \to 0} 3$, and this is just 3 as there is no h dependence.

So, for any a, if f(x) = 3x+5 we have f'(a) = 3. We often use the same variable for f' and f, so we would write f'(x) = 3. Where is this 3 coming from?

Instantaneous Speed for Linear Functions

Let’s take f(x) = 3x + 5 and calculate the instantaneous speed at x=a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

So $f'(a) = \lim_{h \to 0} \frac{(3(a+h)+5)-(3a+5)}{h}$ = $\lim_{h \to 0} \frac{(3a+3h+5)-(3a+5)}{h}$ = $\lim_{h \to 0} \frac{3h}{h}$.

What is $\lim_{h \to 0} \frac{3h}{h}$? It is $\lim_{h \to 0} 3$, and this is just 3 as there is no h dependence.

So, for any a, if f(x) = 3x+5 we have f’(a) = 3. We often use the same variable for f’ and f, so we would write f’(x) = 3. Where is this 3 coming from? The coefficient in front of the linear term.

Instantaneous Speed for Linear Functions

More generally take f(x) = c x + d and calculate the instantaneous speed at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

So $f'(a) = \lim_{h \to 0} \frac{(c(x+h)+d)-(cx+d)}{h}$ = $\lim_{h \to 0} \frac{(cx+ch+d)-(cx+d)}{h}$ = $\lim_{h \to 0} \frac{ch}{h}$.

What is $\lim_{h \to 0} \frac{ch}{h}$? It is $\lim_{h \to 0} c$, and this is just c as there is no h dependence.

Thus if f(x) = cx + d then f'(x) = c.

What functions should we study next?

Instantaneous Speed for Quadratic Functions

Let's take f(x) = $3x^2$+ 5x + 2 and calculate the instantaneous speed at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's look at the numerator:

f(x+h) =

Instantaneous Speed for Quadratic Functions

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Let's look at the numerator:

f(x+h) = $3(x+h)^2$ + 5(x+h) + 2 =

Instantaneous Speed for Quadratic Functions

Let's take $f(x) = 3x^2 + 5x + 2$ and calculate the instantaneous speed at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's look at the numerator:

$f(x+h) = 3(x+h)^2 + 5(x+h) + 2 = 3(\mathbf{1}x^2 + \mathbf{2}hx + \mathbf{1}h^2) + 5(\mathbf{1}x+\mathbf{1}h) + 2$

$\quad\quad\quad = 3x^2 + 6hx + 3h^2 + 5x + 5h + 2$

$f(x) \quad = 3x^2 + 5x + 2$

So $f(x+h) – f(x) = 6hx + 3h^2 + 5h$.

Note the coefficients from Pascal's Triangle…..

Instantaneous Speed for Quadratic Functions

Let's take $f(x) = 3x^2 + 5x + 2$ and calculate the instantaneous speed at x.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's look at the numerator: $f(x+h) – f(x) = 6hx + 3h^2 + 5h$.

Thus

$$f'(x) = \lim_{h \to 0} \frac{6hx+3h^2+5h}{h} = \lim_{h \to 0} (6x + 3h + 5) = 6x + 5.$$

So how do we get from $f(x) = 3x^2 + 5x + 2$ to $f'(x) = 6x + 5$?

Instantaneous Speed for Quadratic Functions

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So how do we get from f(x) = $3x^2$+ 5x + 2 to f'(x) = 6x + 5?

The 6x could be 3 times 2, 3 is the coefficient of x^2 and 2 is the power, and note that the power of x has decreased by 1.

Similarly the 5 could be 5 times 1, where 5 is the coefficient of x and 1 is the power, and note the power of x has decreased by 1.

Instantaneous Speed for Quadratic Functions

If we take f(x) = $ax^2 + bx + c$ and calculate the instantaneous speed at x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h},$$

we find f'(x) = 2ax + b.

We saw if f(x) = ax + b that f'(x) = a.

What would you guess for f(x) = $ax^3 + bx^2 + cx + d$?

Instantaneous Speed for Quadratic Functions

If we take f(x) = ax^2 + bx + c and calculate the instantaneous speed at x,

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We saw if f(x) = ax + b that f’(x) = a.

What would you guess for f(x) = ax^3 + bx^2 + cx + d?

Answer: f’(x) = 3ax^2 + 2bx + c.

Instantaneous Speed for Polynomials

If we take f(x) = ax^n and calculate the instantaneous speed at x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

we find f'(x) = nax^{n-1}.

What is the key ingredient to find f(x+h) = a$(x+h)^n$?

Answer:

Instantaneous Speed for Quadratic Functions

If we take f(x) = ax^2 + bx + c and calculate the instantaneous speed at x,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

we find f'(x) = 2ax + b.

We saw if f(x) = ax + b that f'(x) = a.

What would you guess for f(x) = $ax^3 + bx^2$ + cx + d?

Answer: f'(x) = $3ax^2$ + 2bx + c.

Could now do a few polynomials to test your understanding….

Why do we care?

We can use the instantaneous speed to approximate the function.

We showed if $f(x) = 3x^2 + 5x + 2$ then $f'(x) = 6x + 5$.

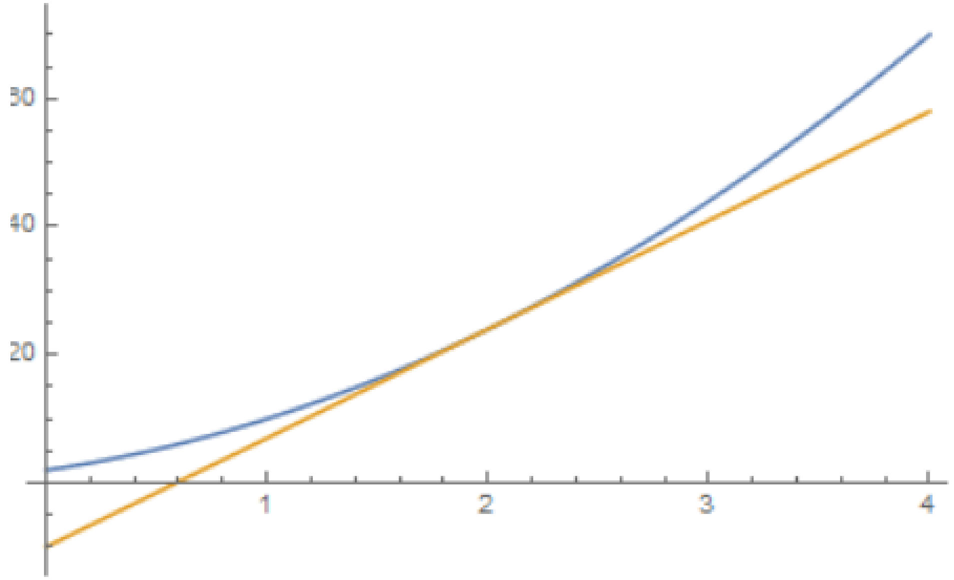
Consider the point x=2. Have $f(2) = 12 + 10 + 2 = 24$.

The instantaneous speed there is $f'(2) = 12 + 5 = 17$.

We can draw the tangent line at this point, using point-slope.

Point: $(2, f(2)) = (2, 24)$ and slope $m = f'(2) = 17$.

Thus line is $y - 24 = 17(x-2)$ or $y = 17x - 10$.



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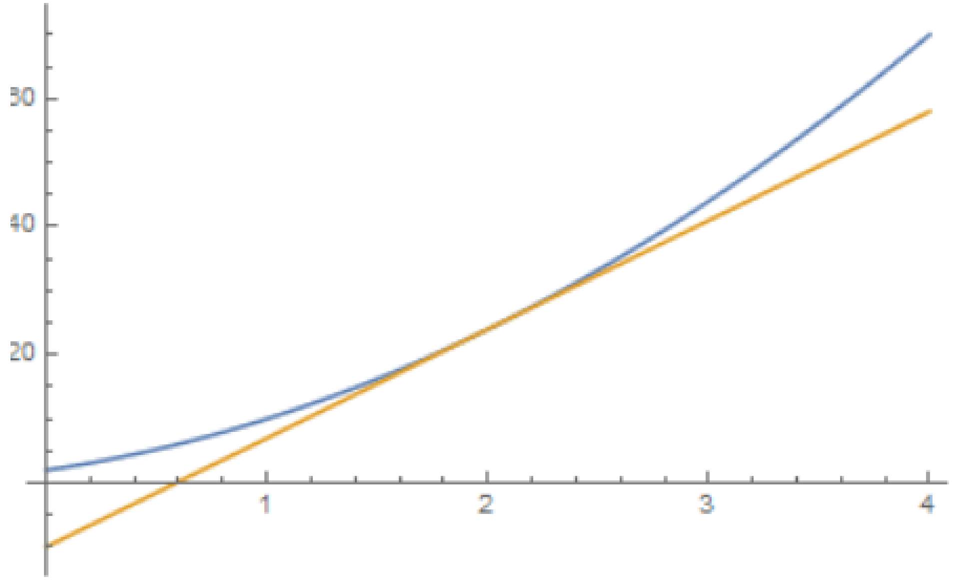


Are you surprised the tangent line is a good approximation near x=2? Why?

PART IV:
Divide and Conquer versus Newton’s Method

Finding roots

Much of math is about solving equations.

Example: polynomials:

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.
- Cubic, quartic: formulas exist in terms of coefficients; not for quintic and higher.

In general cannot find exact solution, how to estimate?

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{3\times 2^{1/3}a}\right\},\right.$$

$$\left\{x \to -\frac{b}{3a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\},$$

$$\left.\left\{x \to -\frac{b}{3a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\}\right\}$$

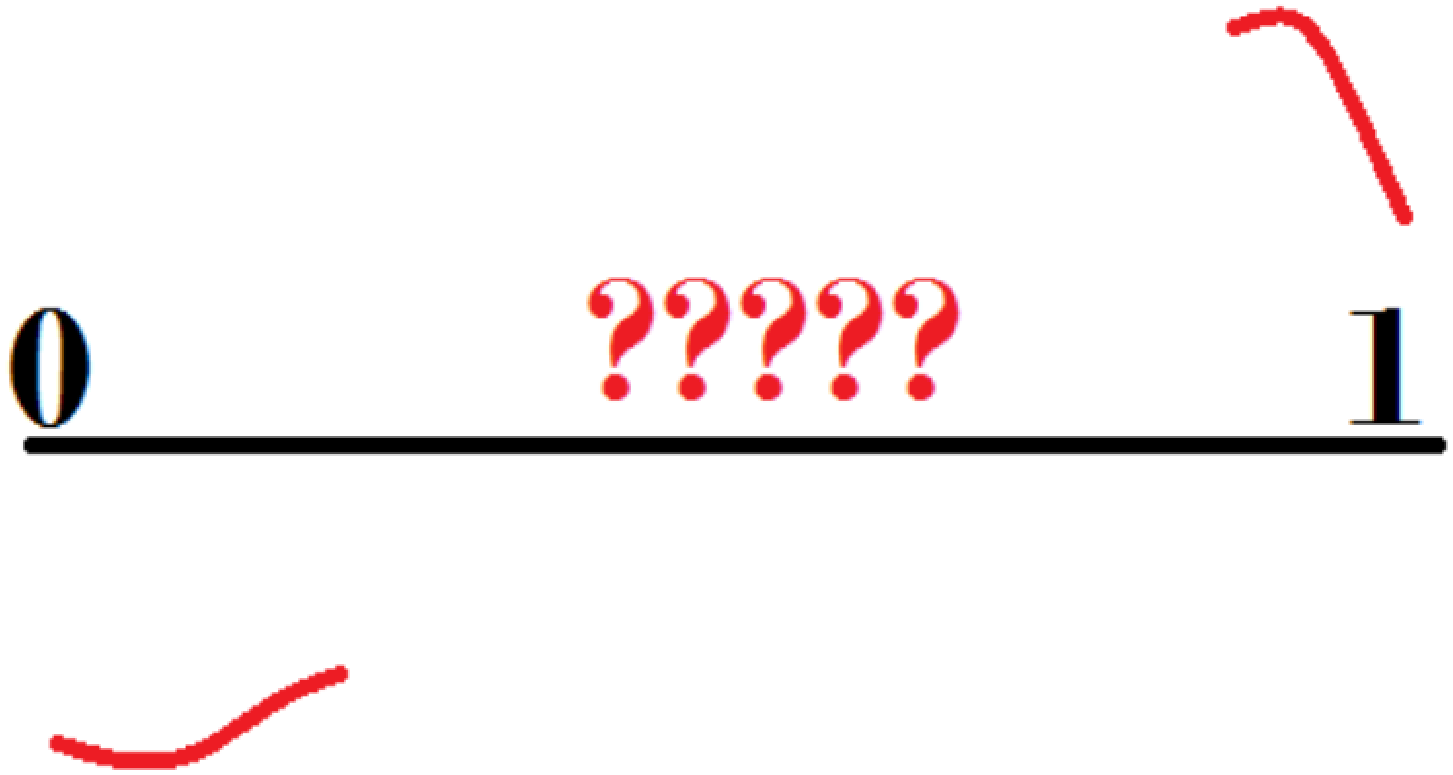
One of the solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
```

$$
\Bigg\{\Bigg\{x \to -\frac{b}{4\,a} - \frac{1}{2}\sqrt{\left(\frac{b^2}{4\,a^2} - \frac{2\,c}{3\,a} + \left(2^{1/3}\left(c^2 - 3\,b\,d + 12\,a\,e\right)\right)\Big/\left(3\,a\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3}\right) + \frac{1}{3 \times 2^{1/3}\,a}\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3}\right)}
$$

$$
- \frac{1}{2}\sqrt{\Bigg(\frac{b^2}{2\,a^2} - \frac{4\,c}{3\,a} - \left(2^{1/3}\left(c^2 - 3\,b\,d + 12\,a\,e\right)\right)\Big/\left(3\,a\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3}\right) - \frac{1}{3 \times 2^{1/3}\,a}\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3} - \left(-\frac{b^3}{a^3} + \frac{4\,b\,c}{a^2} - \frac{8\,d}{a}\right)\Big/\left(4\sqrt{\left(\frac{b^2}{4\,a^2} - \frac{2\,c}{3\,a} + \left(2^{1/3}\left(c^2 - 3\,b\,d + 12\,a\,e\right)\right)\Big/\left(3\,a\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3}\right) + \frac{1}{3 \times 2^{1/3}\,a}\left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e + \sqrt{-4\left(c^2 - 3\,b\,d + 12\,a\,e\right)^3 + \left(2\,c^3 - 9\,b\,c\,d + 27\,a\,d^2 + 27\,b^2\,e - 72\,a\,c\,e\right)^2}\right)^{1/3}\right)}\right)\Bigg)}\Bigg\},
$$

Divide and Conquer



Divide and Conquer

Divide and Conquer

Assume f is continuous, $f(a) < 0 < f(b)$. Then f has a root between a and b. To find, look at the sign of f at the midpoint $f\left(\frac{a+b}{2}\right)$; if sign positive look in $[a, \frac{a+b}{2}]$ and if negative look in $[\frac{a+b}{2}, b]$. Lather, rinse, repeat.

Notation: [a, b] means the interval from a to b: it is all x such that a ≤ x ≤ b. Thus [0,1] is all real numbers from 0 to 1.

Divide and Conquer

Divide and Conquer

Assume f is continuous, $f(a) < 0 < f(b)$. Then f has a root between a and b. To find, look at the sign of f at the midpoint $f\left(\frac{a+b}{2}\right)$; if sign positive look in $[a, \frac{a+b}{2}]$ and if negative look in $[\frac{a+b}{2}, b]$. Lather, rinse, repeat.

Example:

- $f(0) = -1, f(1) = 3$, look at $f(.5)$.
- $f(.5) = 2$, so look at $f(.25)$.
- $f(.25) = -.4$, so look at $f(.375)$.

Divide and Conquer (continued)

How fast? Every 10 iterations uncertainty decreases by a factor of $2^{10} = 1024 \approx 1000$.

Thus 10 iterations essentially give three decimal digits.

		f(x) = x^2 - 3, sqrt(3)		1.732051		
n	left	right	f(left)	f(right)	left error	right error
1	1	2	-2	1	0.732051	-0.26795
2	1.5	2	-0.75	1	0.232051	-0.26795
3	1.5	1.75	-0.75	0.0625	0.232051	-0.01795
4	1.625	1.75	-0.35938	0.0625	0.107051	-0.01795
5	1.6875	1.75	-0.15234	0.0625	0.044551	-0.01795
6	1.71875	1.75	-0.0459	0.0625	0.013301	-0.01795
7	1.71875	1.734375	-0.0459	0.008057	0.013301	-0.00232
8	1.726563	1.734375	-0.01898	0.008057	0.005488	-0.00232
9	1.730469	1.734375	-0.00548	0.008057	0.001582	-0.00232
10	1.730469	1.732422	-0.00548	0.001286	0.001582	-0.00037

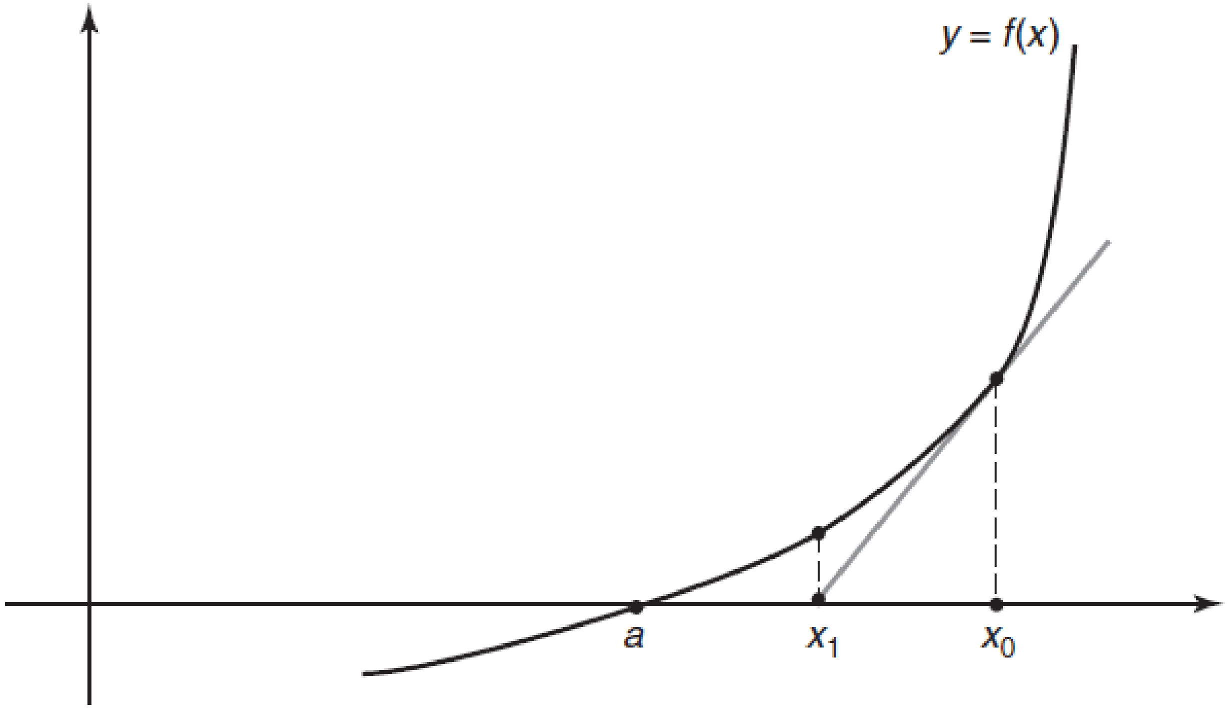
Figure: Approximating $\sqrt{3} \approx 1.7320508075688772935274463415$.

Newton's Method

Newton's Method

Assume f is continuous and differentiable. We generate a sequence hopefully converging to the root of $f(x) = 0$ as follows. Given x_n, look at the tangent line to the curve $y = f(x)$ at x_n; it has slope $f'(x_n)$ and goes through $(x_n, f(x_n))$ and gives line $y - f(x_n) = f'(x_n)(x - x_n)$. This hits the x-axis at $y = 0, x = x_{n+1}$, and yields $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Newton's Method

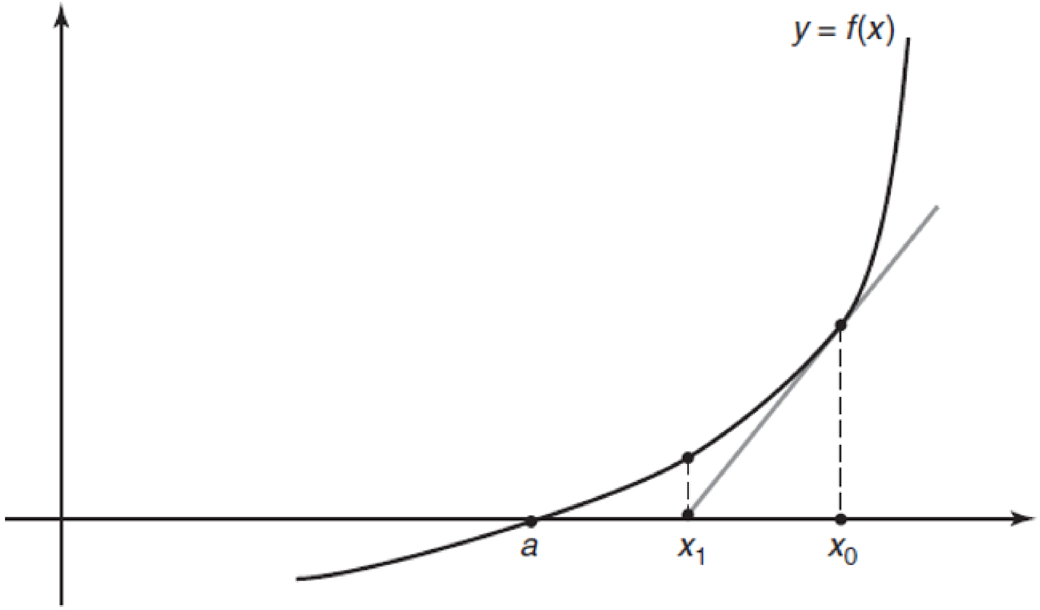


Newton's Method: Finding the next guess

Say have f(x) = $x^2 - 3$

Want to solve f(x) = 0

Roots are



Newton's Method: Finding the next guess

Say have $f(x) = x^2 - 3$

Want to solve $f(x) = 0$

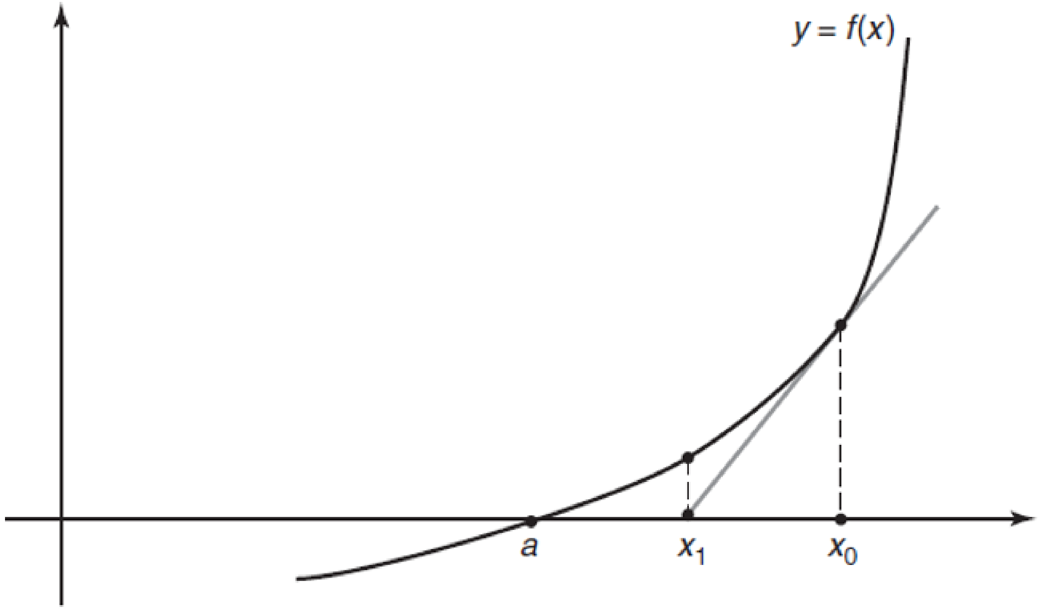
Roots are $\sqrt{3}\ and\ -\sqrt{3}$.

But what are these numbers?

How can we approximate them?

Idea is to replace the quadratic curve y = f(x) with a straight line.

Go from first guess to second, then shampoo math: lather, rinse, repeat.

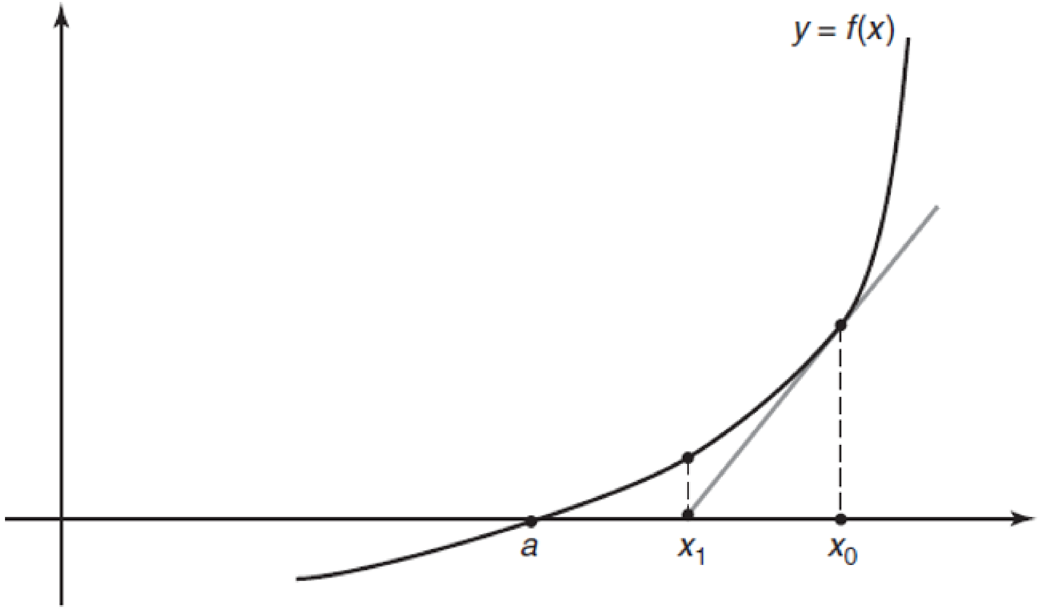


Newton's Method: Finding the next guess

Solve $f(x) = x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

Initial guess $x_0 = 2$.

If we plug in x=2 we get f(2) =



Newton's Method: Finding the next guess

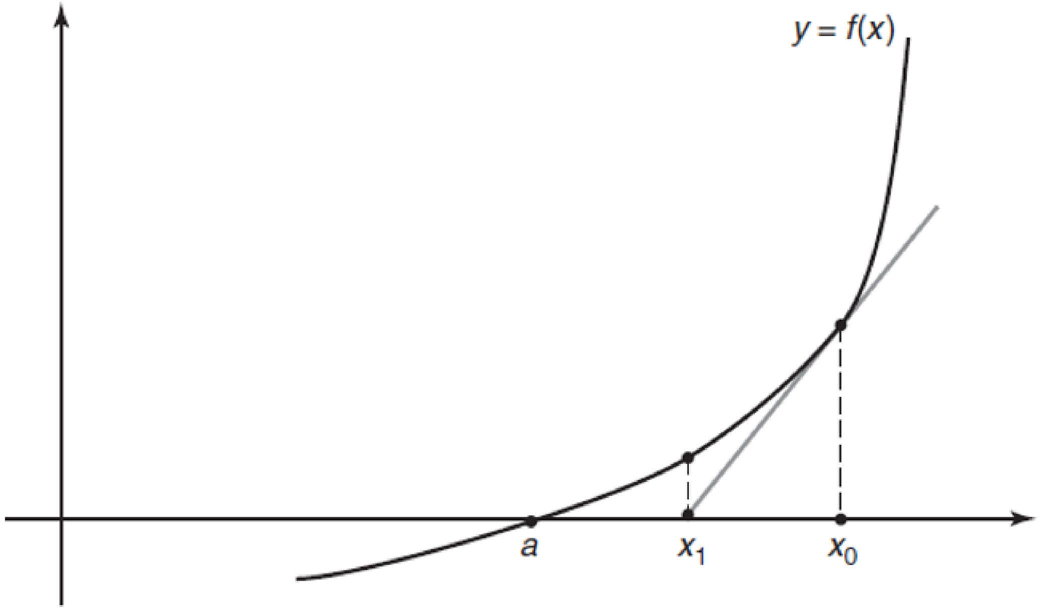
Solve f(x) = $x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

Initial guess $x_0 = 2$.

If we plug in x=2 we get f(2) = 1.

This is NOT zero, so we have NOT found the root.

What is f'(x)? It is f'(x) =



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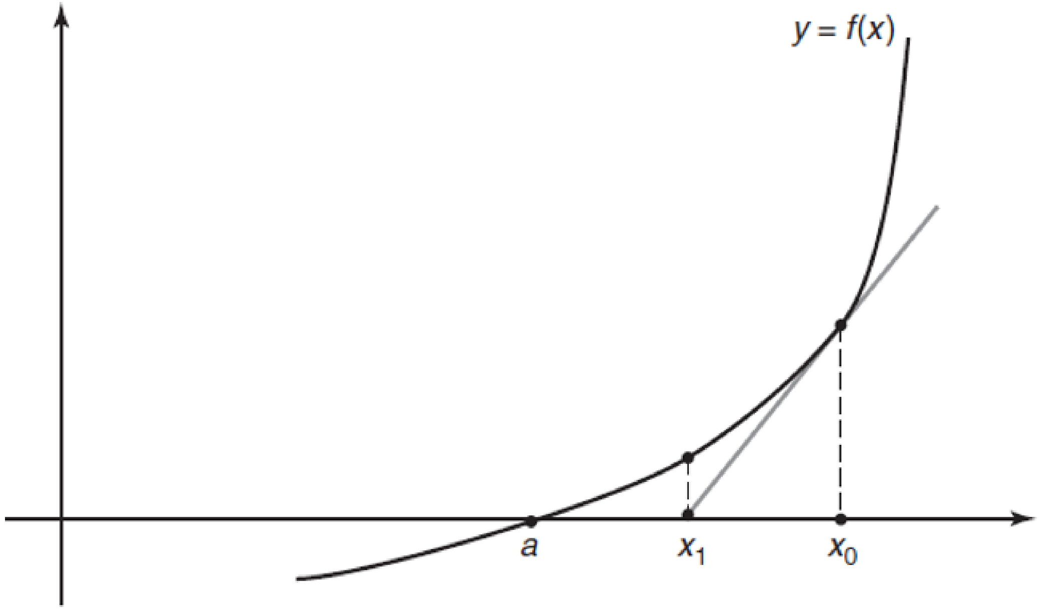
Thus what is the instantaneous speed at x=2? It is f'(2) =



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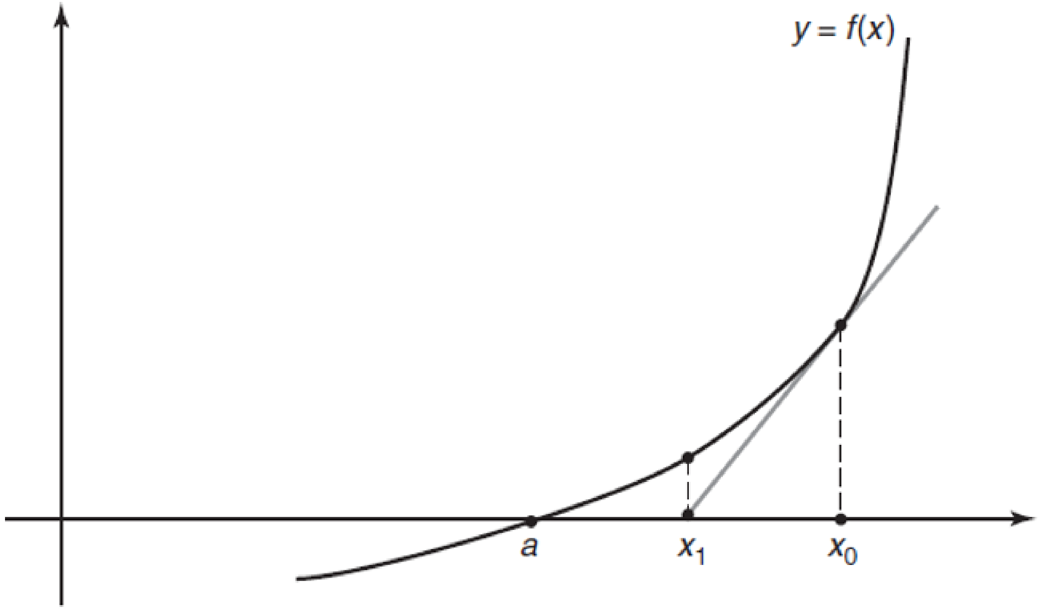
What is f'(x)? It is f'(x) = 2x .

Thus what is the instantaneous speed at x=2? It is f'(2) = 4.

Now we find the tangent line here.



Newton's Method: Finding the next guess

Solve f(x) = $x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

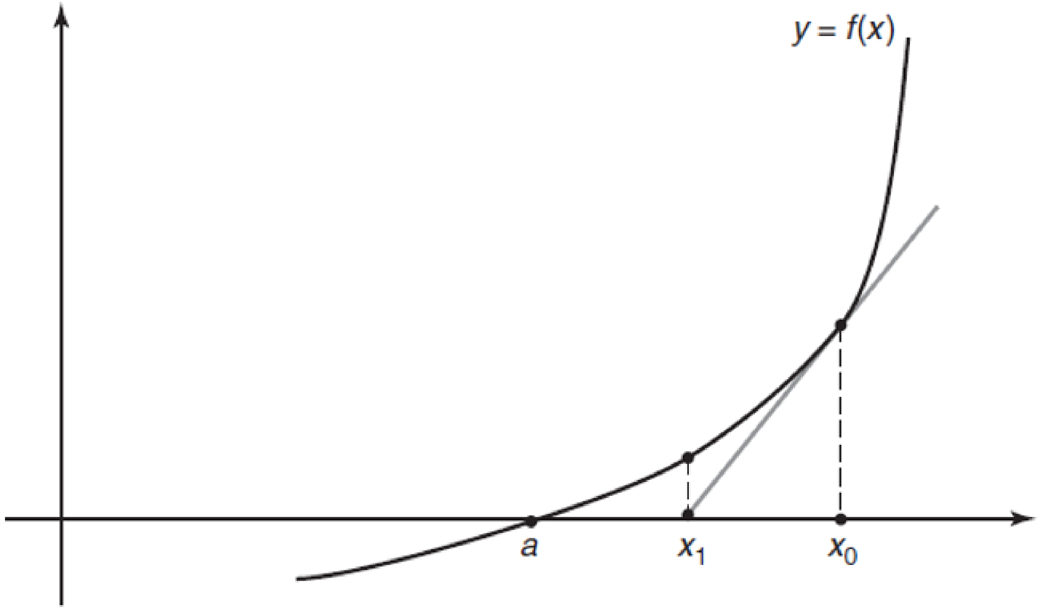
Initial guess $x_0 = 2$, f(2) = 1, f'(2) = 4.

Use Point-Slope to get line.

Point: (2,f(2)) = (2,1)

Slope: m = f'(2) = 4.

Equation:



Newton's Method: Finding the next guess

Solve $f(x) = x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

Initial guess $x_0 = 2$, $f(2) = 1$, $f'(2) = 4$.

Use Point-Slope to get line.

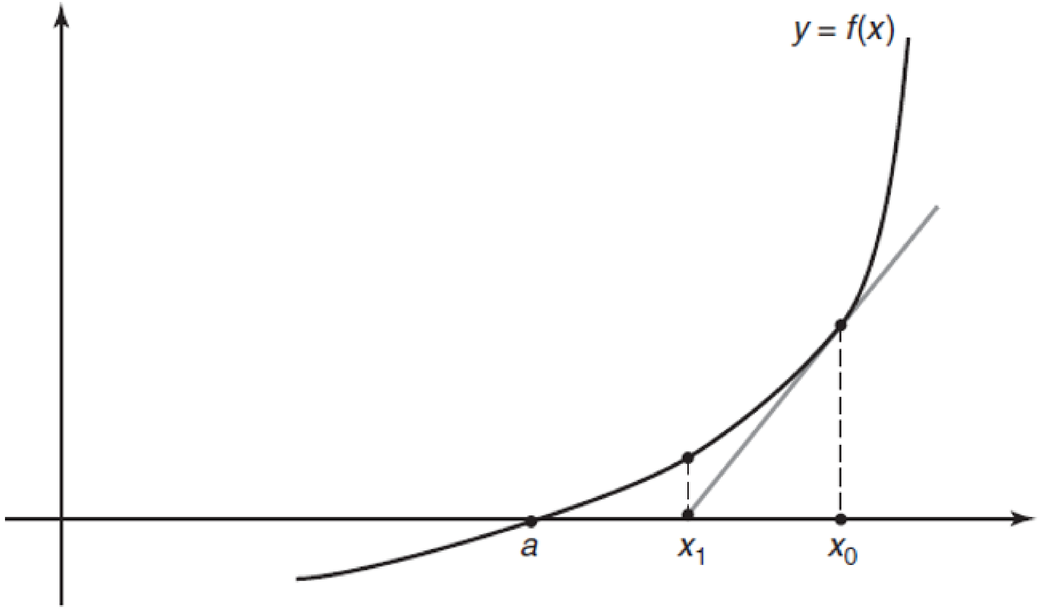
Point: $(2,f(2)) = (2,1)$

Slope: $m = f'(2) = 4$.

Equation: $y - 1 = 4(x-2)$.

Simplify to $y = 4x-7$.

Where does this line hit the x-axis?



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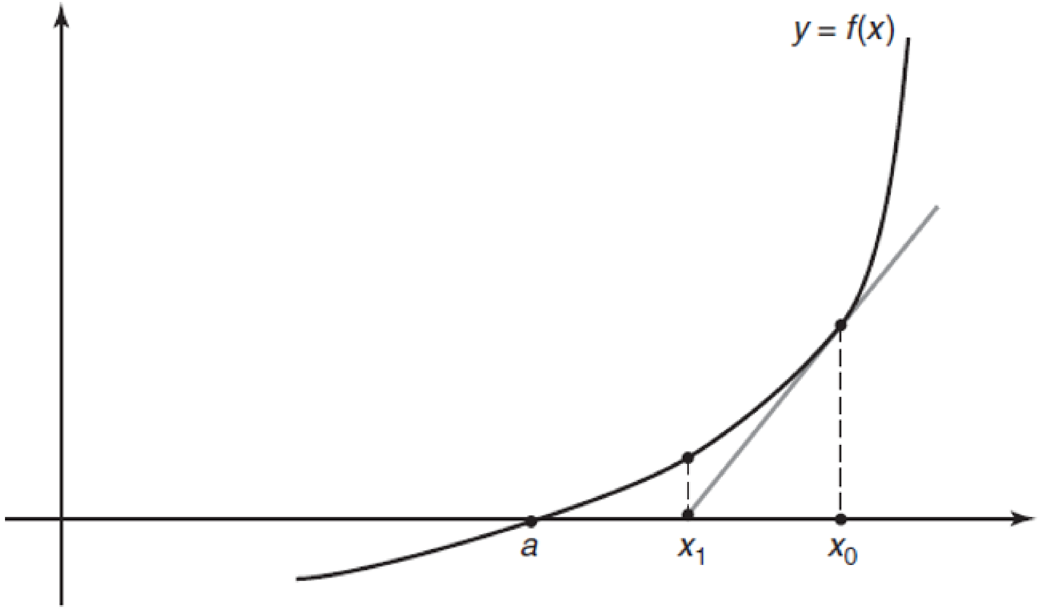
Where does this line hit the x-axis?

0 = 4x-7 so 4x = 7 so x = 7/4 = 1.75.



$\sqrt{3} = 1.7320508075688772935274463415058723 6$

Newton's Method: Finding the next guess

Solve $f(x) = x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

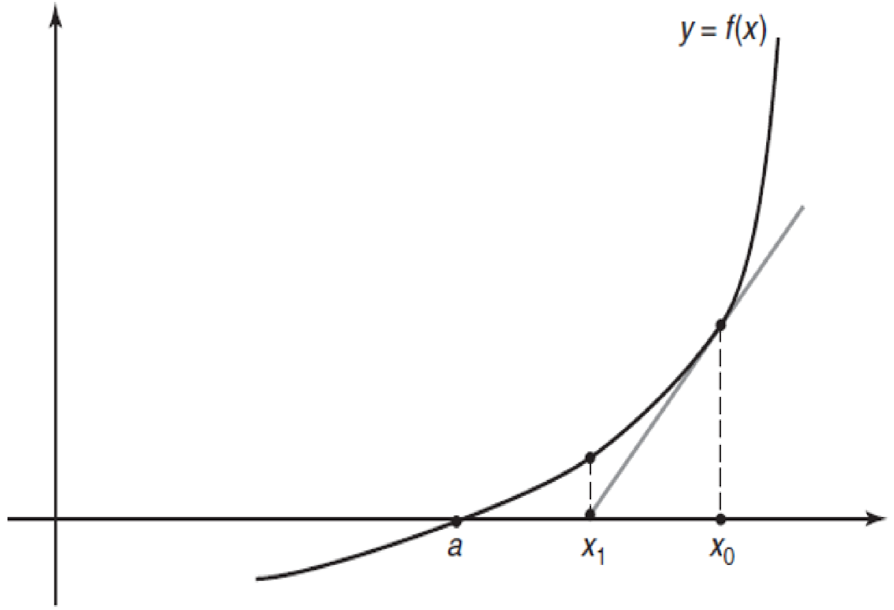
We had $x_0 = 2$ our initial guess.

New guess for root is $x_1 = 7/4 = 1.75$.

$\sqrt{3} = 1.7320508075688772935274463415058723$6

Not terrible.

Try again. Use $x_1 = 7/4$, and find a NEW tangent line….



Newton's Method: Finding the next guess

Solve $f(x) = x^2 - 3 = 0$, roots are $\sqrt{3}\ and\ -\sqrt{3}$.

New guess for root is $x_1 = 7/4 = 1.75$.

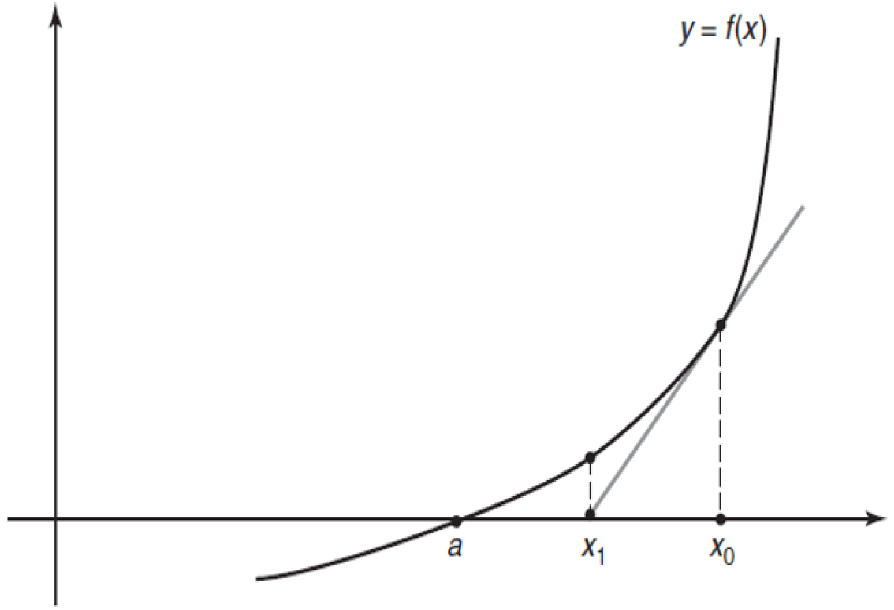
Point: (7/4, f(7/4)) = (7/4, 49/16-3)

Slope: m = f'(7/4) = 7/2 as f'(x) = 2x.

Simplify:

Point is (7/4, 1/16), slope is 7/2.

As y-coordinate almost 0 see CLOSE to the root….



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Solve $f(x) = x^2 - 3 = 0$, roots are $\sqrt{3}$ *and* $-\sqrt{3}$.

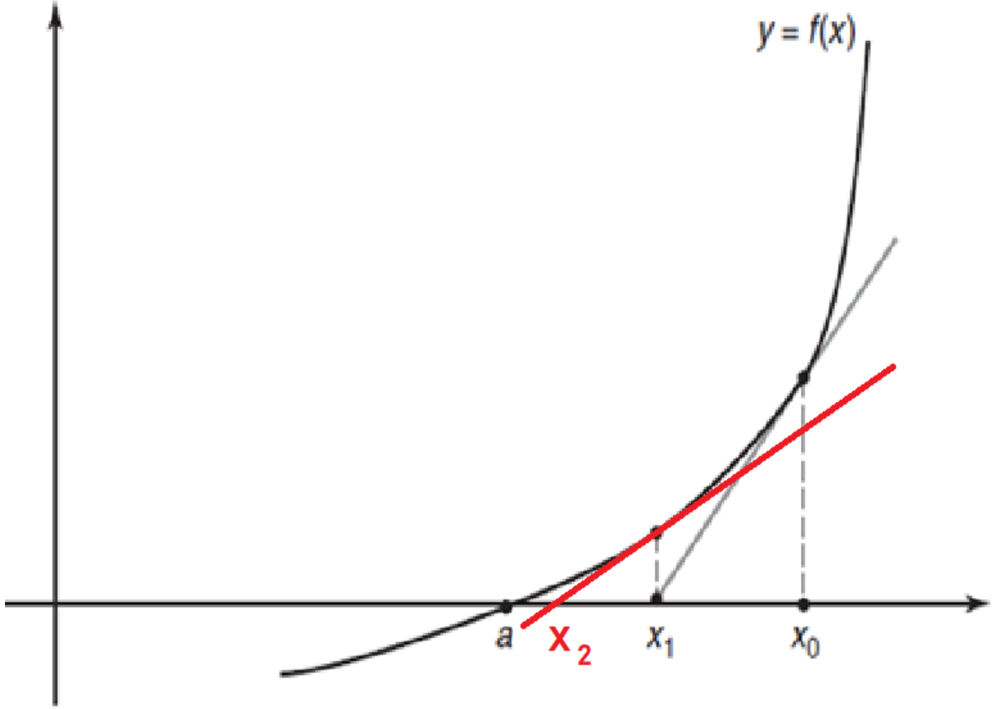
New guess for root is $x_1 = 7/4 = 1.75$.

Point: $(7/4, f(7/4)) = (7/4, 1/16)$

Slope: $m = f'(7/4) = 7/2$ as $f'(x) = 2x$.

Line: $y - 1/16 = (7/2)(x - 7/4)$.

Where does this hit the x-axis?



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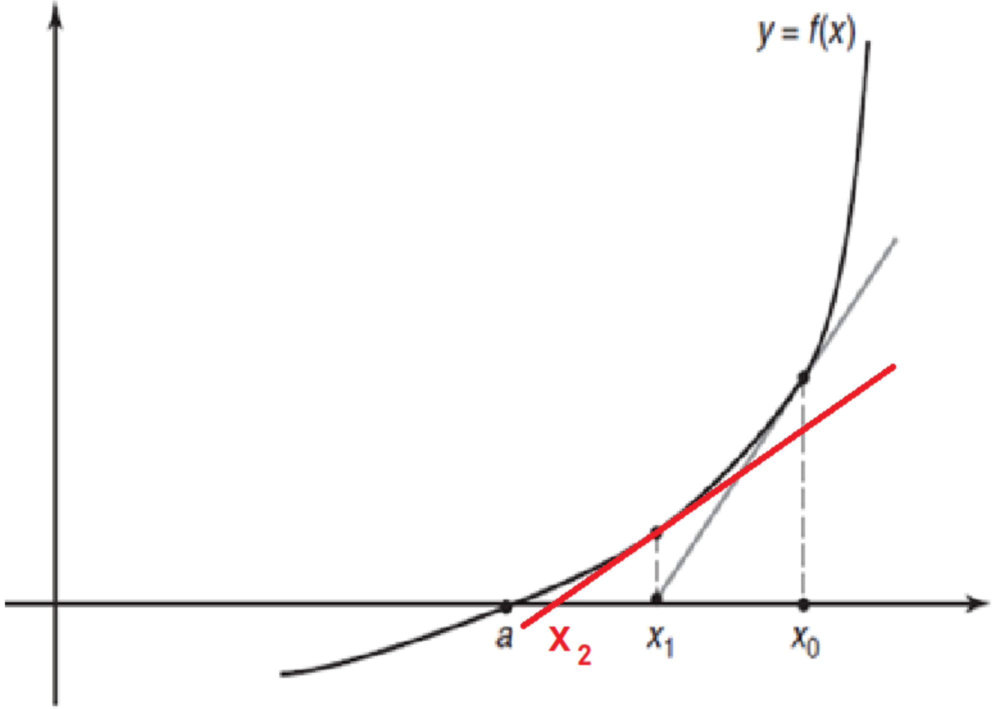
Slope: $m = f'(7/4) = 7/2$ as $f'(x) = 2x$.

Line: $y - 1/16 = (7/2)(x - 7/4)$.

Where does this hit the x-axis? At $y = 0$.

Get $-1/16 = (7/2)x - 49/8$



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Point: (7/4, f(7/4)) = (7/4, 1/16)

Slope: m = f'(7/4) = 7/2 as f'(x) = 2x.

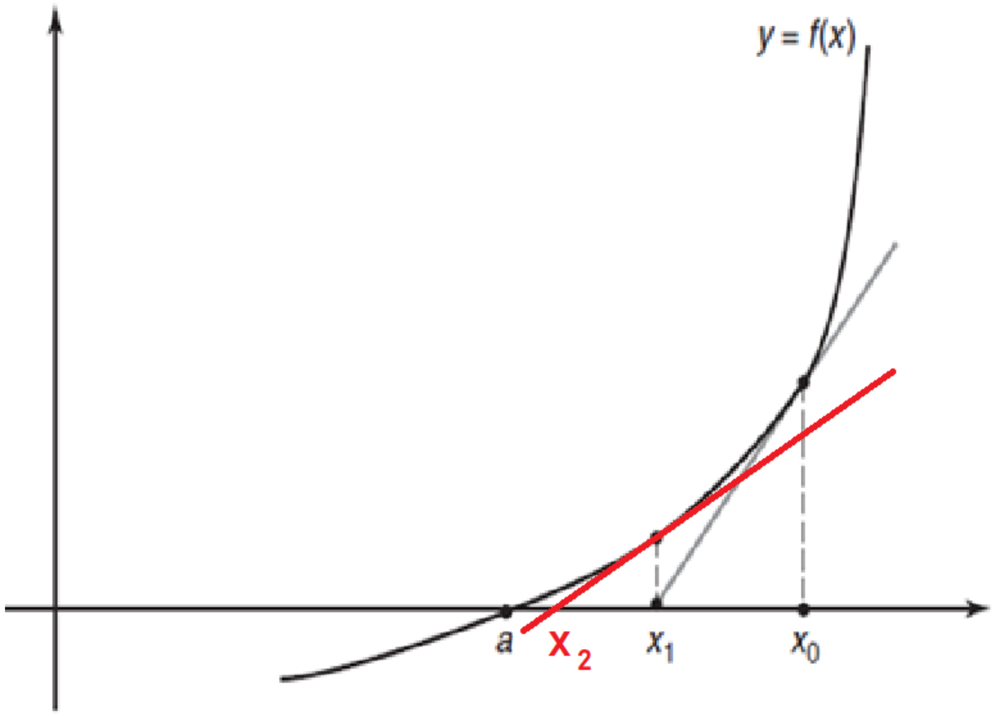
Line: y – 1/16 = (7/2)(x – 7/4).

Where does this hit the x-axis? At y = 0.

Get -1/16 = (7/2)x – 49/8

So (7/2)x = 97/16, or x = 97/56

So next guess x_2 is 97/56 or about 1.7321428571.



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Newton's Method: Finding the next guess

Solve f(x) = $x^2 - 3 = 0$, roots are $\sqrt{3}$ *and* $-\sqrt{3}$.

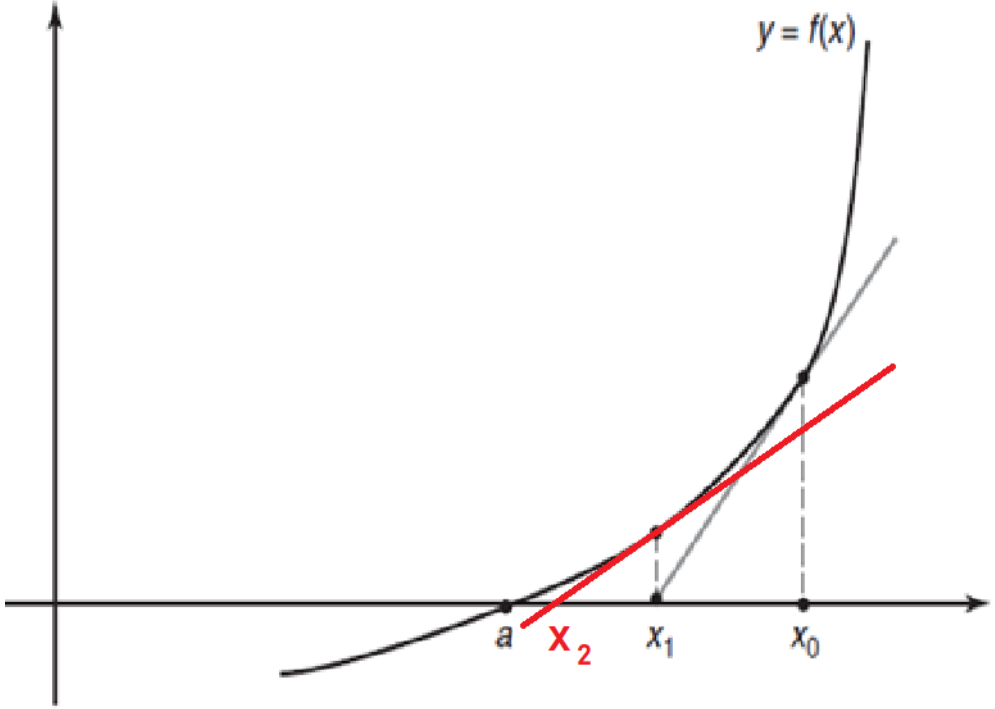
New guess for root is x_2 = 97/56 or about 1.7321428571.

We can keep doing this.

We get a sequence of points $x_1, x_2, x_3, \ldots$.

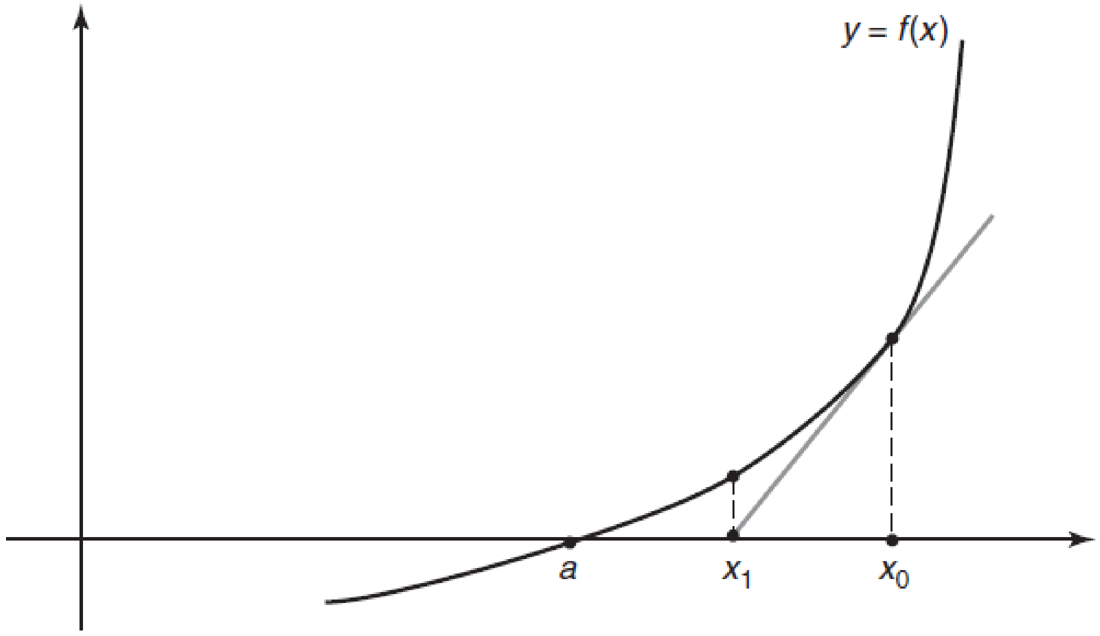
Do these converge to $\sqrt{3}$? Looks like it!

Doing some algebra we can come up with an explicit formula for x_{n+1} in terms of x_n.



$\sqrt{3} = 1.7320508075688772935274463415058723 6$

Newton's Method



For example, $f(x) = x^2 - 3$ after algebra get $x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$.

Newton's Method

n	x[n]	1.0 x[n]	Sqrt[3] - x[n]
0	2	2.000000000000000000000000000000000000000	-0.2679491924311227064725536584941276330 57
1	$\frac{7}{4}$	1.7500000000000000000000000000000000000000	-0.01794919243112270647255365849412763305 7
2	$\frac{97}{56}$	1.7321428571428572062984585500008945167065	-0.000092049573979849329696515636984775914
3	$\frac{18817}{10864}$	1.7320508100147276042690691610914655029774	$-2.4458502469732900355191644519 08\times10^{-9}$

Sqrt[3] = 1.7320508075688772935274463415058723669428

x[5] = 1.7320508075688772935274463415058723678037

x[4] = 1.73205080756887729525435394607217191423|51

$$\sqrt{3} = 1.732050807568877293527446341505872366\mathbf{6}9428$$

$$x_5 = 1.732050807568877293527446341505872367\mathbf{7}8037$$

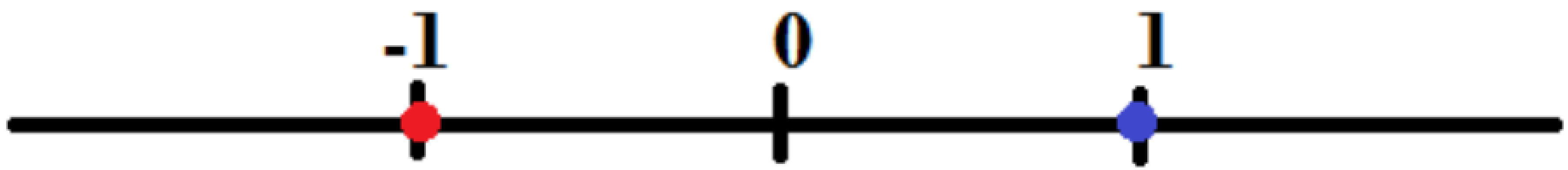
$$x_5 = \frac{1002978273411373057}{579069776145402304}.$$

Newton Method: $x^2 - 3 = 0$

Consider $x^2 - 1 = (x - 1)(x + 1) = 0$.

Roots are 1, -1; if we start at a point x_0 do we approach a root? If so which?

Recall $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$.



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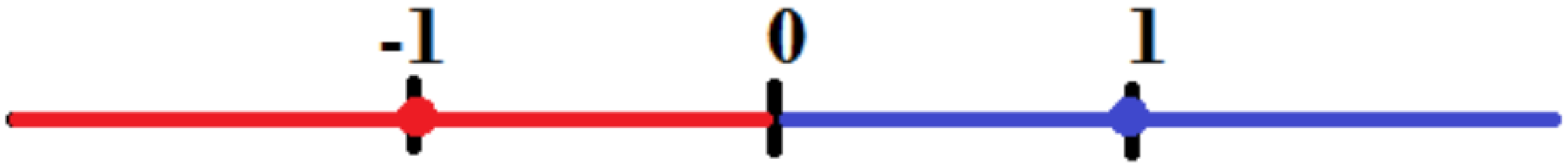


Newton Fractal: $x^3 - 1 = 0$: https://www.youtube.com/watch?v=ZsFixqGFNRc

What are the roots to $x^3 - 1 = 0$?

Here comes Complex Numbers!
$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}.$

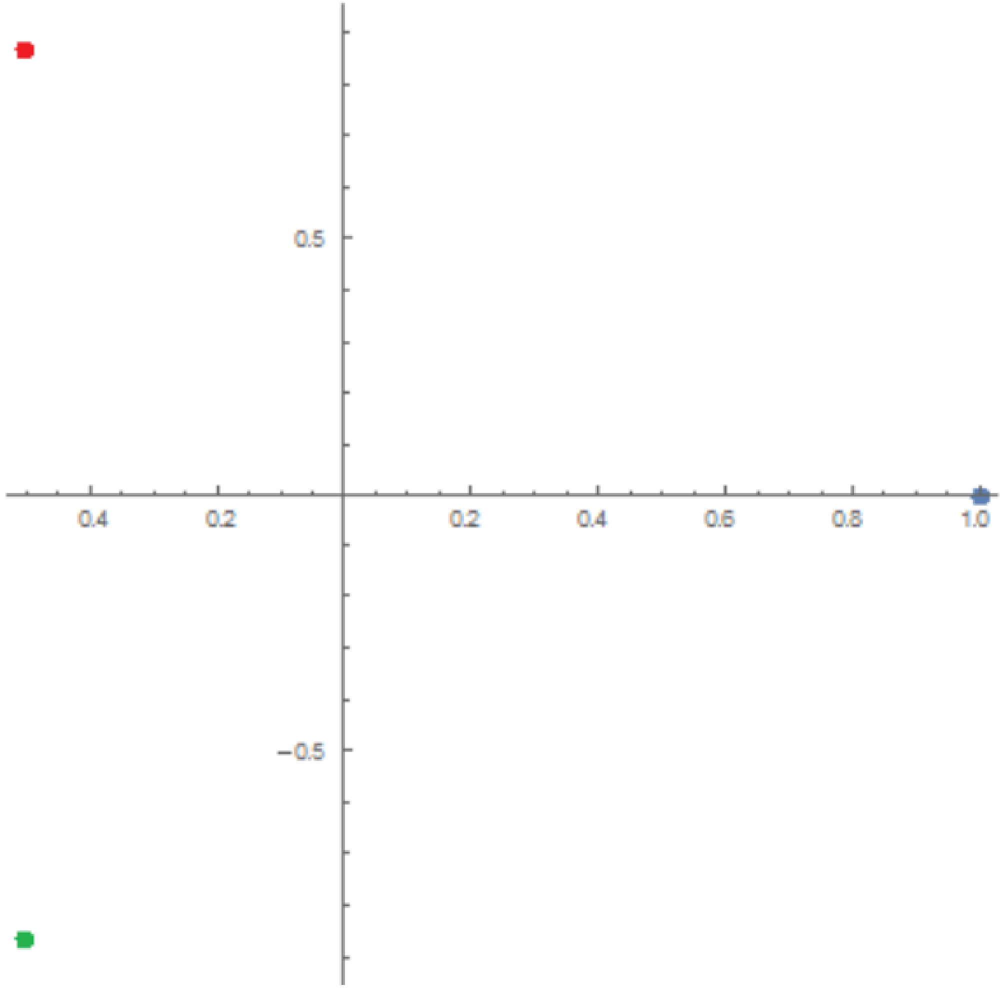
$$
\begin{aligned}
x^3 - 1 &= (x-1)(x^2 + x + 1) \\
&= (x-1)\cdot\left(x - \frac{-1+\sqrt{1^2 - 4\cdot 1\cdot 1}}{2}\right)\cdot\left(x - \frac{-1-\sqrt{1^2 - 4\cdot 1\cdot 1}}{2}\right) \\
&= (x-1)\cdot\left(x - \frac{-1+\sqrt{-3}}{2}\right)\cdot\left(x - \frac{-1-\sqrt{-3}}{2}\right) \\
&= (x-1)\cdot\left(x - \frac{-1+i\sqrt{3}}{2}\right)\cdot\left(x - \frac{-1-i\sqrt{3}}{2}\right).
\end{aligned}
$$

Roots are 1, $-1/2 + i\sqrt{3}/2$, $-1/2 - i\sqrt{3}/2$.

https://www.youtube.com/watch?v=ZsFixqGFNRc

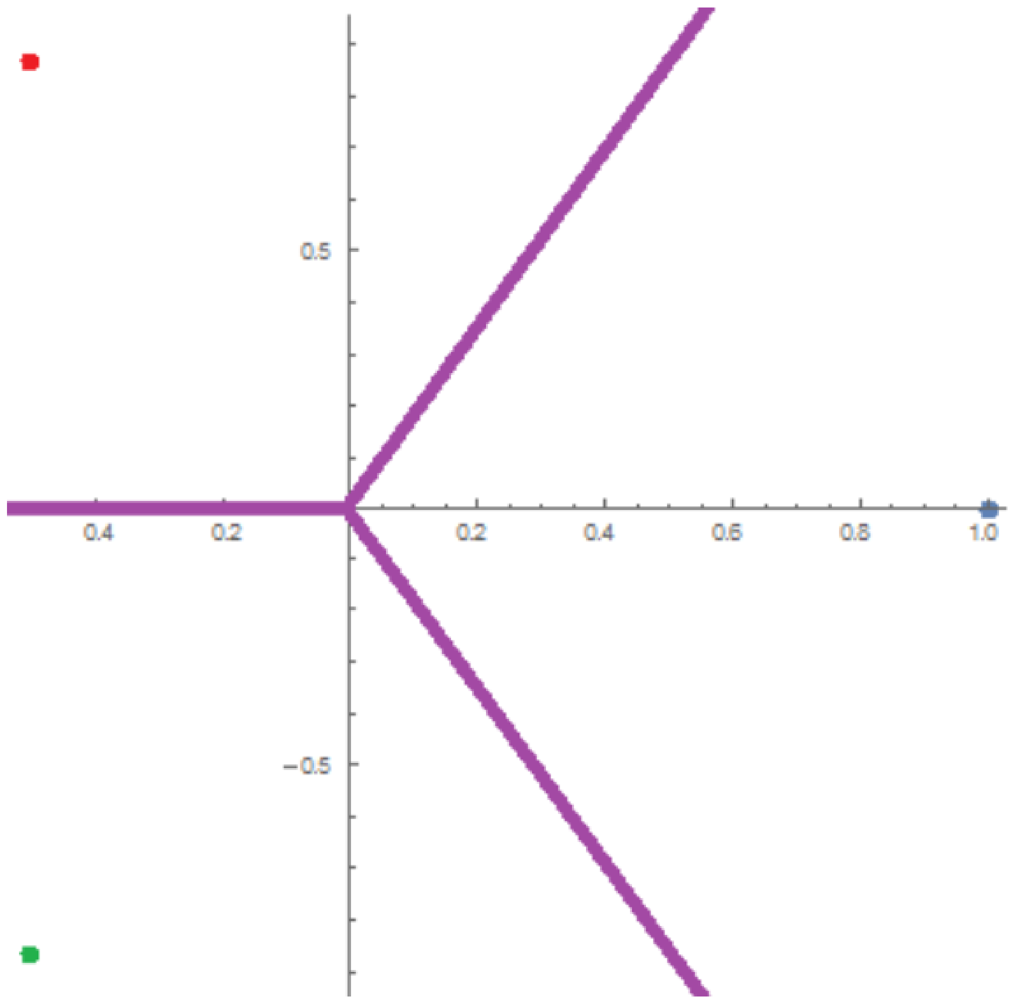
Newton Fractal: $x^3 - 1 = 0$: https://www.youtube.com/watch?v=ZsFixqGFNRc

If start at (x, y), what root do you iterate to?

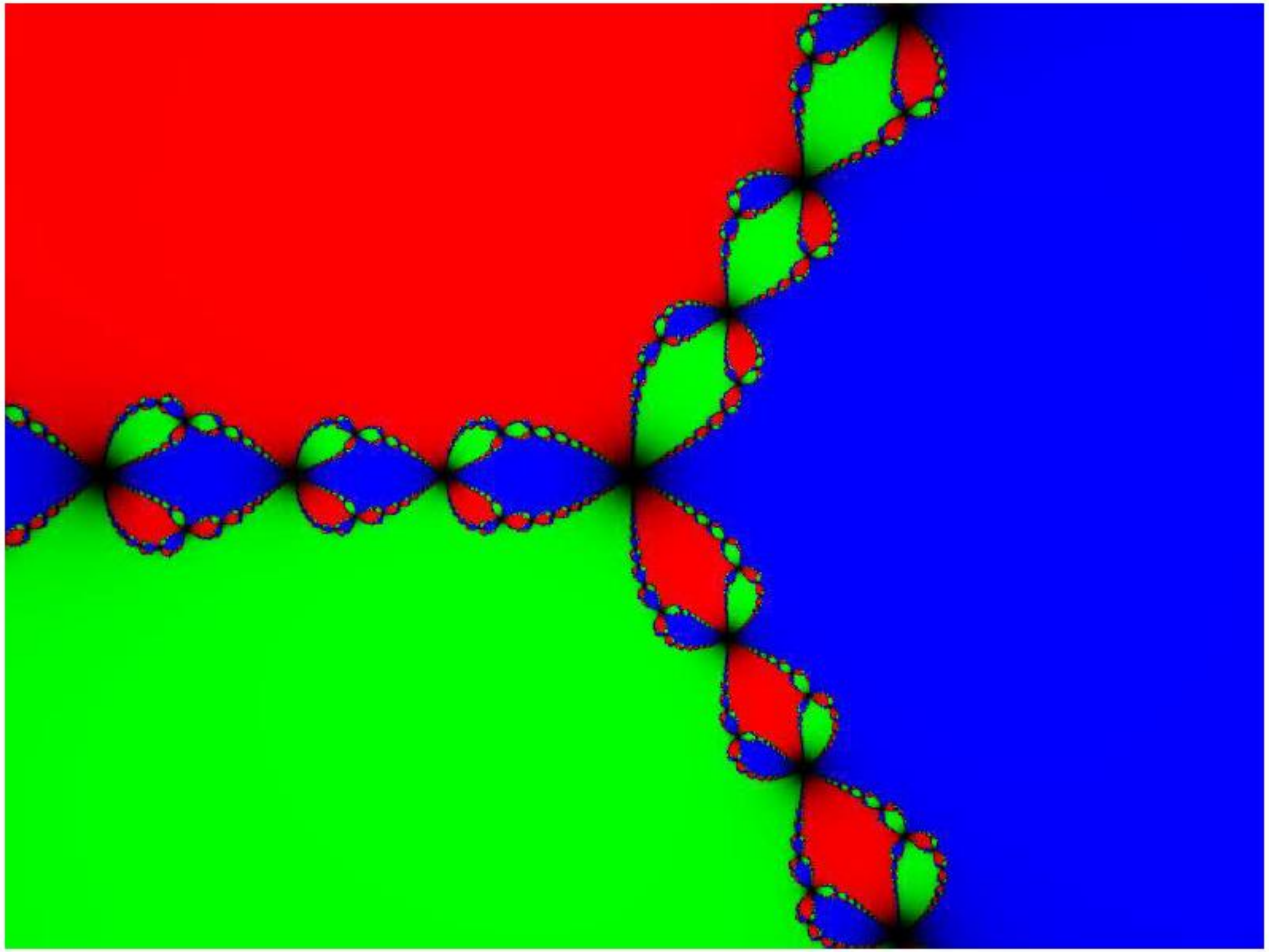


Newton Fractal: $x^3 - 1 = 0$: https://www.youtube.com/watch?v=ZsFixqGFNRc

If start at (x, y), what root do you iterate to? Guess



Newton Fractal: $x^3 - 1 = 0$: https://www.youtube.com/watch?v=ZsFixqGFNRc



https://www.youtube.com/watch?v=ZsFixqGFNRc

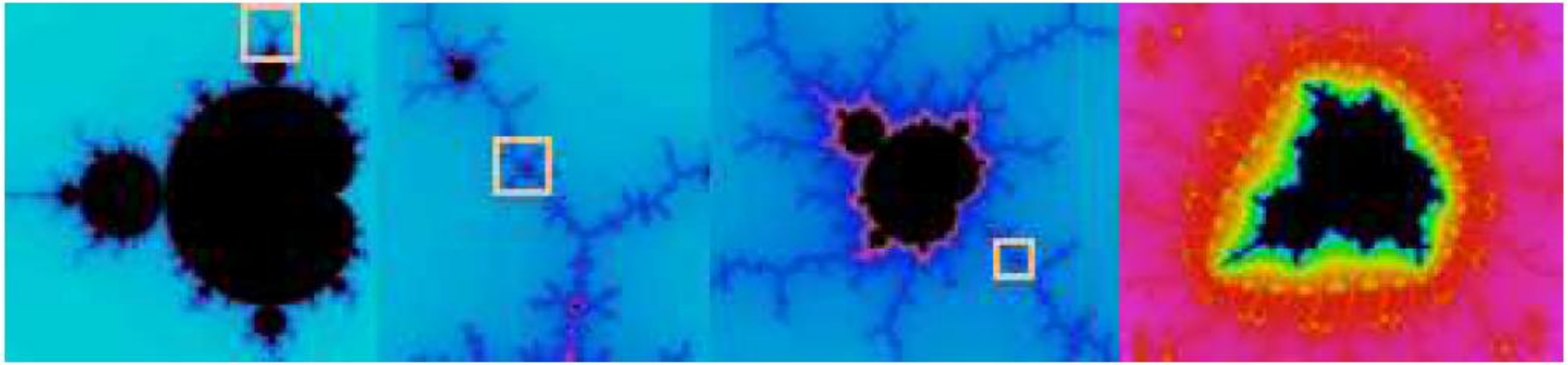
Mandelbrot Set: https://www.youtube.com/watch?v=0jGaio87u3A

Definition: Set of all complex numbers $c = x + iy$ ($i = \sqrt{-1}$) such that if $f_c(u) = u^2 + c$ then the sequence

$$z_1 = f_c(0), \quad z_2 = f_c(z_1) = f_c(f_c(0)), \quad \cdots, \quad z_{n+1} = f_c(z_n)$$

remains bounded as $n \to \infty$.

Zooming in....



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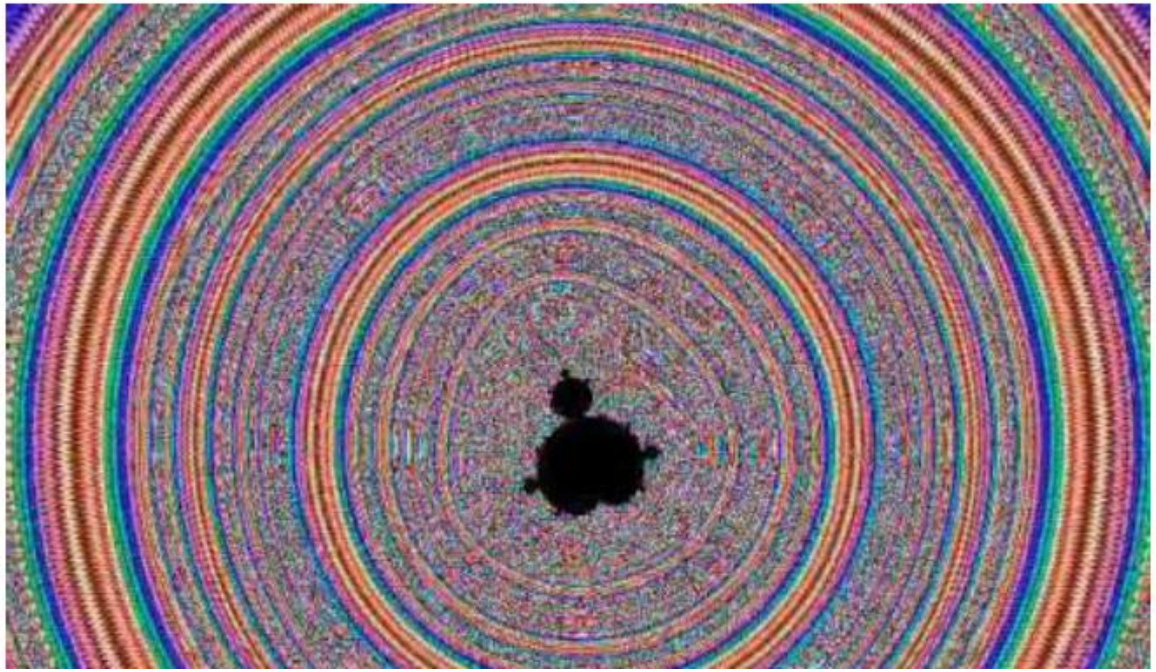
Extreme zoom!



GOOD LINKS

https://www.youtube.com/watch?v=0jGaio87u3A

https://www.youtube.com/watch?v=9j2yV1nLCEI

https://www.youtube.com/watch?v=ZsFixqGFNRc

https://www.youtube.com/watch?v=PD2XgQOyCCk

https://www.youtube.com/watch?v=vfteiiTfE0c

Why do we care?

If such small changes can lead to such wildly different behavior, what happens when we try to solve the equations governing our world?